Analytical Classical Dynamics:

An intermediate level course

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1 Introduction

1.1 Intended audience

These lecture notes outline a single semester course intended for upper division undergraduates.

1.2 Major sources

The textbooks which I have consulted most frequently whilst developing course material are:

Analytical mechanics, G.R. Fowles (Holt, Rinehart & Winston, New York NY, 1977).

Classical dynamics of particles and systems, 5th Edition, S.T. Thornton, and J.B. Marion (Brooks/Cole—Thomson Learning, Belmont CA, 2004).

Analytical mechanics, G.R. Fowles, and G.L. Cassiday, 7th Edition (Brooks/Cole—Thomson Learning, Belmont CA, 2005).

1.3 Scope of course

The scope of this course is indicated by its title, "Analytical Classical Dynamics". Taking the elements of the title in reverse order, "Dynamics" is the study of the *motions* of the various objects in the world around us. A mathematical theory of dynamics is an axiomatic system, ultimately based on a few fundamental laws, which can be used to both understand and predict these motions. By "Classical", we understand that the theory of motion which we are going to use in our investigation of dynamics is that first published by Isaac Newton in 1687. We now know that this theory is only *approximately* true. The theory breaks down when the velocities of the objects under investigation approach the speed of light, and must be replaced by Einstein's special theory of *relativity*. The theory also breaks down

on the atomic and subatomic scales, and must be replaced by *quantum mechanics*. In this course, we shall neglect both relativistic and quantum effects entirely. It follows that we must restrict our investigation to the motions of large (compared to an atom) slow (compared to the speed of light) objects. Fortunately, however, most of the motions which we observe in the world around us fall into this class. Finally, by "Analytical", we understand that we shall only consider those types of motion whose governing differential equations can be solved via standard analytic techniques. In practice, this means that the governing equations must be *linear* in nature, since our ability to solve nonlinear differential equations analytically is very limited. Fortunately, a wide range of the observed motions in the world around us are governed, either exactly or approximately, by linear differential equations. Unfortunately, there is one very interesting type of motion which is definitely *not* governed by linear differential equations—namely, *chaotic motion*. It is impossible to make a meaningful investigation of chaotic motion without resorting to *numerical methods* for solving the associated differential equations. Such methods lie well beyond the scope of this course. Consequently, we shall not be discussing chaotic motion.

1.4 Outline of course

This course is organized as follows. Section 2 is a review of those elements of vector algebra and vector calculus which are needed in classical dynamics. Section 3 discusses the fundamental aspects of Newton's theory of motion. Section 4 investigates one-dimensional motion, including damped oscillatory motion. Section 5 discusses multi-dimensional motion. Section 6 investigates the motions of the Planets in the Solar System. Section 7 discusses two-body problems such as scattering. Section 8 investigates motion in non-inertial reference frames. Section 9 discusses the rotation of rigid bodies. Section 10 investigates Lagrangian dynamics. Section 11 is devoted to Hamiltonian dynamics. Finally, Sect. 12 investigates coupled oscillations.

2 Vectors

2.1 Introduction

In this section, we shall give a brief outline of those aspects of vector algebra and vector calculus which are needed to investigate classical dynamics.

This section is largely based on my undergraduate lecture notes from a course given by Dr. Stephen Gull at the University of Cambridge.

2.2 Vector algebra

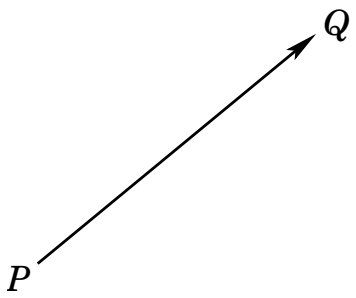

Figure 1:

In applied mathematics, physical quantities are (predominately) represented by two distinct classes of objects. Some quantities, denoted *scalars*, are represented by *real numbers*. Others, denoted *vectors*, are represented by directed line elements in space: *e.g.*, $\overrightarrow{PQ}$ (see Fig. 1). Note that line elements (and, therefore, vectors) are movable, and do not carry intrinsic position information. In fact, vectors just possess a magnitude and a direction, whereas scalars possess a magnitude but no direction. By convention, vector quantities are denoted by bold-faced characters (*e.g.*, $\mathbf{a}$) in typeset documents, and by underlined characters (*e.g.*, $\underline{a}$) in long-hand. Vectors can be added together, but the *same units* must be used, just like in scalar addition. Vector addition can be represented using a parallelogram: $\overrightarrow{PR} = \overrightarrow{PQ} + \overrightarrow{QR}$ (see Fig. 2). Suppose that $\mathbf{a} \equiv \overrightarrow{PQ} \equiv \overrightarrow{SR}$, $\mathbf{b} \equiv \overrightarrow{QR} \equiv \overrightarrow{PS}$, and $\mathbf{c} \equiv \overrightarrow{PR}$. It is clear from Fig. 2 that vector addition is *commutative*: *i.e.*, $\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$. It can

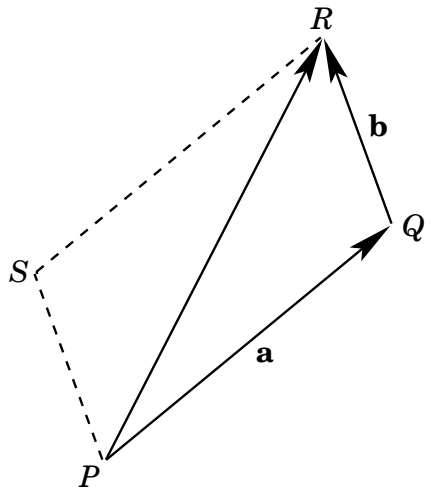

Figure 2:

also be shown that the *associative* law holds: *i.e.*, $\mathbf{a}+(\mathbf{b}+\mathbf{c})=(\mathbf{a}+\mathbf{b})+\mathbf{c}$.

There are two approaches to vector analysis. The *geometric* approach is based on line elements in space. The *coordinate* approach assumes that space is defined by Cartesian coordinates, and uses these to characterize vectors. In physics, we generally adopt the second approach, because it is far more convenient than the first.

In the coordinate approach, a vector is denoted as the row matrix of its components along each of the Cartesian axes (the x-, y-, and z-axes, say):

$$\mathbf{a}\equiv(a_x,\,a_y,\,a_z). \tag{2.1}$$

Here, a_x is the x-coordinate of the "head" of the vector minus the x-coordinate of its "tail." If $\mathbf{a}\equiv(a_x,a_y,a_z)$ and $\mathbf{b}\equiv(b_x,b_y,b_z)$ then vector addition is defined

$$\mathbf{a}+\mathbf{b}\equiv(a_x+b_x,\,a_y+b_y,\,a_z+b_z). \tag{2.2}$$

If $\mathbf{a}$ is a vector and n is a scalar then the product of a scalar and a vector is defined

$$n\,\mathbf{a}\equiv(n\,a_x,\,n\,a_y,\,n\,a_z). \tag{2.3}$$

It is clear that vector algebra is *distributive* with respect to scalar multiplication: *i.e.*, $n\,(\mathbf{a}+\mathbf{b})=n\,\mathbf{a}+n\,\mathbf{b}$.

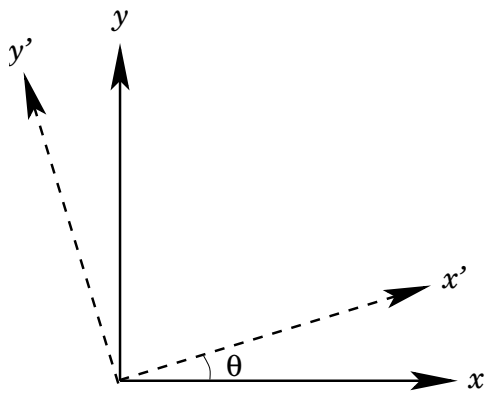

Figure 3:

Unit vectors can be defined in the x-, y-, and z-directions as $\mathbf{e}_x \equiv (1,0,0)$, $\mathbf{e}_y \equiv (0,1,0)$, and $\mathbf{e}_z \equiv (0,0,1)$. Any vector can be written in terms of these unit vectors:

$$\mathbf{a} = a_x\,\mathbf{e}_x + a_y\,\mathbf{e}_y + a_z\,\mathbf{e}_z. \tag{2.4}$$

In mathematical terminology, three vectors used in this manner form a *basis* of the vector space. If the three vectors are mutually perpendicular then they are termed *orthogonal basis vectors*. However, any set of three non-coplanar vectors can be used as basis vectors.

Examples of vectors in physics are displacements from an origin,

$$\mathbf{r} = (x,\, y,\, z), \tag{2.5}$$

and velocities,

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \lim_{\delta t \to 0} \frac{\mathbf{r}(t+\delta t) - \mathbf{r}(t)}{\delta t}. \tag{2.6}$$

Suppose that we transform to a new orthogonal basis, the x'-, y'-, and z'-axes, which are related to the x-, y-, and z-axes via a rotation through an angle θ around the z-axis (see Fig. 3). In the new basis, the coordinates of the general displacement $\mathbf{r}$ from the origin are (x', y', z'). These coordinates are related to the previous coordinates via the transformation:

$$x' = x\,\cos\theta + y\,\sin\theta, \tag{2.7}$$

$$y' = -x\,\sin\theta + y\,\cos\theta, \tag{2.8}$$

$$z' = z. \tag{2.9}$$

We do not need to change our notation for the displacement in the new basis. It is still denoted **r**. The reason for this is that the magnitude and direction of **r** are *independent* of the choice of basis vectors. The coordinates of **r** *do* depend on the choice of basis vectors. However, they must depend in a very specific manner [*i.e.*, Eqs. (2.7)–(2.9)] which preserves the magnitude and direction of **r**.

Since any vector can be represented as a displacement from an origin (this is just a special case of a directed line element), it follows that the components of a general vector **a** must transform under rotation through an angle θ about the z-axis in an analogous manner to Eqs. (2.7)–(2.9). Thus,

$$a_{x'} = a_x \cos\theta + a_y \sin\theta, \tag{2.10}$$

$$a_{y'} = -a_x \sin\theta + a_y \cos\theta, \tag{2.11}$$

$$a_{z'} = a_z, \tag{2.12}$$

with similar transformation rules for rotation about the x- and y-axes. In the coordinate approach, Eqs. (2.10)–(2.12) constitute the *definition* of a vector. The three quantities (a_x, a_y, a_z) are the components of a vector provided that they transform under rotation like Eqs. (2.10)–(2.12). Conversely, (a_x, a_y, a_z) *cannot* be the components of a vector if they do not transform like Eqs. (2.10)–(2.12). Scalar quantities are *invariant* under transformation. Thus, the individual components of a vector (a_x, say) are real numbers, but they are *not* scalars. Displacement vectors, and all vectors derived from displacements, automatically satisfy Eqs. (2.10)–(2.12). There are, however, other physical quantities which have both magnitude and direction, but which are not obviously related to displacements. We need to check carefully to see whether these quantities are vectors.

2.3 The scalar product

A scalar quantity is invariant under all possible rotational transformations. The individual components of a vector are not scalars because they change under transformation. Can we form a scalar out of some combination of the components of one, or more, vectors? Suppose that we were to define the "ampersand" product,

$$\mathbf{a} \,\&\, \mathbf{b} = a_x\, b_y + a_y\, b_z + a_z\, b_x = \text{scalar number}, \tag{2.13}$$

for general vectors $\mathbf{a}$ and $\mathbf{b}$. Is $\mathbf{a}\,\&\,\mathbf{b}$ invariant under transformation, as must be the case if it is a scalar number? Let us consider an example. Suppose that $\mathbf{a} = (1,\, 0,\, 0)$ and $\mathbf{b} = (0,\, 1,\, 0)$. It is easily seen that $\mathbf{a}\,\&\,\mathbf{b} = 1$. Let us now rotate the basis through 45° about the z-axis. In the new basis, $\mathbf{a} = (1/\sqrt{2},\, -1/\sqrt{2},\, 0)$ and $\mathbf{b} = (1/\sqrt{2},\, 1/\sqrt{2},\, 0)$, giving $\mathbf{a}\,\&\,\mathbf{b} = 1/2$. Clearly, $\mathbf{a}\,\&\,\mathbf{b}$ is *not* invariant under rotational transformation, so the above definition is a bad one.

Consider, now, the *dot product* or *scalar product*:

$$\mathbf{a}\cdot\mathbf{b} = a_x\, b_x + a_y\, b_y + a_z\, b_z = \text{scalar number}. \tag{2.14}$$

Let us rotate the basis though θ degrees about the z-axis. According to Eqs. (2.10)–(2.12), in the new basis $\mathbf{a}\cdot\mathbf{b}$ takes the form

$$\begin{aligned}\mathbf{a}\cdot\mathbf{b} &= (a_x\,\cos\theta + a_y\,\sin\theta)\,(b_x\,\cos\theta + b_y\,\sin\theta)\\ &\quad +(-a_x\,\sin\theta + a_y\,\cos\theta)\,(-b_x\,\sin\theta + b_y\,\cos\theta) + a_z\, b_z\\ &= a_x\, b_x + a_y\, b_y + a_z\, b_z.\end{aligned} \tag{2.15}$$

Thus, $\mathbf{a}\cdot\mathbf{b}$ *is* invariant under rotation about the z-axis. It can easily be shown that it is also invariant under rotation about the x- and y-axes. Clearly, $\mathbf{a}\cdot\mathbf{b}$ is a true scalar, so the above definition is a good one. Incidentally, $\mathbf{a}\cdot\mathbf{b}$ is the *only* simple combination of the components of two vectors which transforms like a scalar. It is easily shown that the dot product is commutative and distributive:

$$\begin{aligned}\mathbf{a}\cdot\mathbf{b} &= \mathbf{b}\cdot\mathbf{a},\\ \mathbf{a}\cdot(\mathbf{b}+\mathbf{c}) &= \mathbf{a}\cdot\mathbf{b} + \mathbf{a}\cdot\mathbf{c}.\end{aligned} \tag{2.16}$$

The associative property is meaningless for the dot product, because we cannot have $(\mathbf{a}\cdot\mathbf{b})\cdot\mathbf{c}$, since $\mathbf{a}\cdot\mathbf{b}$ is scalar.

We have shown that the dot product $\mathbf{a}\cdot\mathbf{b}$ is coordinate independent. But what is the physical significance of this? Consider the special case where $\mathbf{a} = \mathbf{b}$. Clearly,

$$\mathbf{a}\cdot\mathbf{b} = {a_x}^2 + {a_y}^2 + {a_z}^2 = \text{Length } (OP)^2, \tag{2.17}$$

if $\mathbf{a}$ is the position vector of P relative to the origin O. So, the invariance of $\mathbf{a}\cdot\mathbf{a}$ is equivalent to the invariance of the length, or magnitude, of vector $\mathbf{a}$ under

transformation. The length of vector **a** is usually denoted $|a|$ ("the modulus of a") or sometimes just a, so

$$\mathbf{a} \cdot \mathbf{a} = |a|^2 = a^2. \tag{2.18}$$

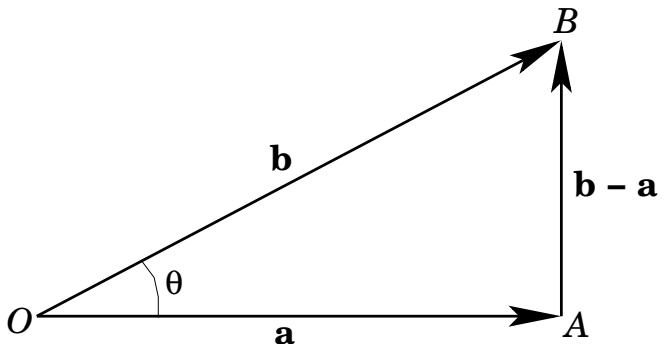

Figure 4:

Let us now investigate the general case. The length squared of AB (see Fig. 4) is

$$(\mathbf{b} - \mathbf{a}) \cdot (\mathbf{b} - \mathbf{a}) = |a|^2 + |b|^2 - 2\,\mathbf{a} \cdot \mathbf{b}. \tag{2.19}$$

However, according to the "cosine rule" of trigonometry,

$$(AB)^2 = (OA)^2 + (OB)^2 - 2\,(OA)\,(OB)\,\cos\theta, \tag{2.20}$$

where (AB) denotes the length of side AB. It follows that

$$\mathbf{a} \cdot \mathbf{b} = |a|\,|b|\,\cos\theta. \tag{2.21}$$

Clearly, the invariance of $\mathbf{a} \cdot \mathbf{b}$ under transformation is equivalent to the invariance of the angle subtended between the two vectors. Note that if $\mathbf{a} \cdot \mathbf{b} = 0$ then either $|a| = 0$, $|b| = 0$, or the vectors **a** and **b** are perpendicular. The angle subtended between two vectors can easily be obtained from the dot product:

$$\cos\theta = \frac{\mathbf{a} \cdot \mathbf{b}}{|a|\,|b|}. \tag{2.22}$$

The work W performed by a constant force **F** moving an object through a displacement **r** is the product of the magnitude of **F** times the displacement in the direction of **F**. If the angle subtended between **F** and **r** is θ then

$$W = |F|\,(|r|\,\cos\theta) = \mathbf{F} \cdot \mathbf{r}. \tag{2.23}$$

The infinitesimal work dW done by a (possibly time-varying) force $\mathbf{F}$ which moves an object through the infinitesimal displacement $d\mathbf{r}$ in the time interval dt is $dW = \mathbf{F}\cdot d\mathbf{r}$. Hence, the instantaneous rate at which the force does work on the object, or the power P, takes the form

$$P = \frac{dW}{dt} = \mathbf{F}\cdot\mathbf{v}, \tag{2.24}$$

where $\mathbf{v} = d\mathbf{r}/dt$ is the object's instantaneous velocity.

2.4 The vector product

We have discovered how to construct a scalar from the components of two general vectors $\mathbf{a}$ and $\mathbf{b}$. Can we also construct a vector which is not just a linear combination of $\mathbf{a}$ and $\mathbf{b}$? Consider the following definition:

$$\mathbf{a}\,\mathrm{x}\,\mathbf{b} = (a_x\,b_x,\, a_y\,b_y,\, a_z\,b_z). \tag{2.25}$$

Is $\mathbf{a}\,\mathrm{x}\,\mathbf{b}$ a proper vector? Suppose that $\mathbf{a} = (1,\,0,\,0)$, $\mathbf{b} = (0,\,1,\,0)$. Clearly, $\mathbf{a}\,\mathrm{x}\,\mathbf{b} = \mathbf{0}$. However, if we rotate the basis through 45° about the z-axis then $\mathbf{a} = (1/\sqrt{2},\,-1/\sqrt{2},\,0)$, $\mathbf{b} = (1/\sqrt{2},\,1/\sqrt{2},\,0)$, and $\mathbf{a}\,\mathrm{x}\,\mathbf{b} = (1/2,\,-1/2,\,0)$. Thus, $\mathbf{a}\,\mathrm{x}\,\mathbf{b}$ does not transform like a vector, because its magnitude depends on the choice of axes. So, above definition is a bad one.

Consider, now, the *cross product* or *vector product*:

$$\mathbf{a}\times\mathbf{b} = (a_y\,b_z - a_z\,b_y,\, a_z\,b_x - a_x\,b_z,\, a_x\,b_y - a_y\,b_x) = \mathbf{c}. \tag{2.26}$$

Does this rather unlikely combination transform like a vector? Let us try rotating the basis through θ degrees about the z-axis using Eqs. (2.10)–(2.12). In the new basis,

$$\begin{aligned} c_{x'} &= (-a_x\,\sin\theta + a_y\,\cos\theta)\,b_z - a_z\,(-b_x\,\sin\theta + b_y\,\cos\theta)\\ &= (a_y\,b_z - a_z\,b_y)\,\cos\theta + (a_z\,b_x - a_x\,b_z)\,\sin\theta\\ &= c_x\,\cos\theta + c_y\,\sin\theta. \end{aligned} \tag{2.27}$$

Thus, the x-component of $\mathbf{a}\times\mathbf{b}$ transforms correctly. It can easily be shown that the other components transform correctly as well. Thus, $\mathbf{a}\times\mathbf{b}$ is a proper vector.

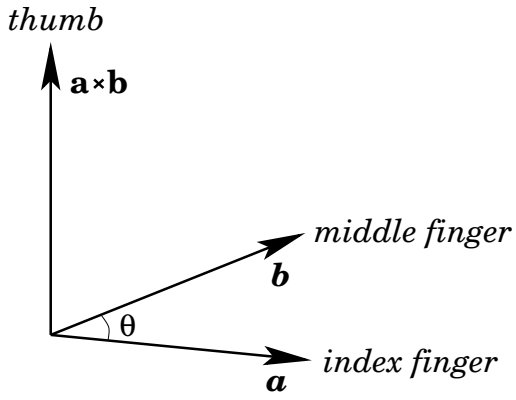

Figure 5:

Incidentally, $\mathbf{a}\times\mathbf{b}$ is the *only* simple combination of the components of two vectors which transforms like a vector (which is non-coplanar with $\mathbf{a}$ and $\mathbf{b}$). The cross product is *anticommutative*,

$$\mathbf{a}\times\mathbf{b} = -\mathbf{b}\times\mathbf{a}, \tag{2.28}$$

distributive,

$$\mathbf{a}\times(\mathbf{b}+\mathbf{c}) = \mathbf{a}\times\mathbf{b}+\mathbf{a}\times\mathbf{c}, \tag{2.29}$$

but is *not* associative:

$$\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) \neq (\mathbf{a}\times\mathbf{b})\times\mathbf{c}. \tag{2.30}$$

The cross product transforms like a vector, which means that it must have a well-defined direction and magnitude. We can show that $\mathbf{a}\times\mathbf{b}$ is *perpendicular* to both $\mathbf{a}$ and $\mathbf{b}$. Consider $\mathbf{a}\cdot\mathbf{a}\times\mathbf{b}$. If this is zero then the cross product must be perpendicular to $\mathbf{a}$. Now

$$\begin{aligned}\mathbf{a}\cdot\mathbf{a}\times\mathbf{b} &= a_x\,(a_y\,b_z - a_z\,b_y) + a_y\,(a_z\,b_x - a_x\,b_z) + a_z\,(a_x\,b_y - a_y\,b_x)\\ &= 0.\end{aligned} \tag{2.31}$$

Therefore, $\mathbf{a}\times\mathbf{b}$ is perpendicular to $\mathbf{a}$. Likewise, it can be demonstrated that $\mathbf{a}\times\mathbf{b}$ is perpendicular to $\mathbf{b}$. The vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{a}\times\mathbf{b}$ form a *right-handed* set, like the unit vectors $\mathbf{e}_x$, $\mathbf{e}_y$, and $\mathbf{e}_z$. In fact, $\mathbf{e}_x\times\mathbf{e}_y = \mathbf{e}_z$. This defines a unique direction for $\mathbf{a}\times\mathbf{b}$, which is obtained from the right-hand rule (see Fig. 5).

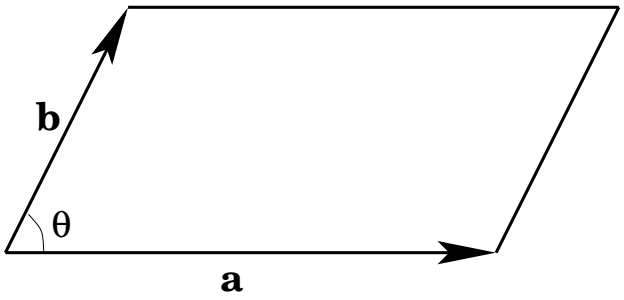

Figure 6:

Let us now evaluate the magnitude of $\mathbf{a}\times\mathbf{b}$. We have

$$\begin{aligned}(\mathbf{a}\times\mathbf{b})^2 &= (a_y\,b_z-a_z\,b_y)^2+(a_z\,b_x-a_x\,b_z)^2+(a_x\,b_z-a_y\,b_x)^2\\ &= (a_x^{\,2}+a_y^{\,2}+a_z^{\,2})\,(b_x^{\,2}+b_y^{\,2}+b_z^{\,2})-(a_x\,b_x+a_y\,b_y+a_z\,b_z)^2\\ &= |a|^2\,|b|^2-(\mathbf{a}\cdot\mathbf{b})^2\\ &= |a|^2\,|b|^2-|a|^2\,|b|^2\,\cos^2\theta=|a|^2\,|b|^2\,\sin^2\theta.\end{aligned}\tag{2.32}$$

Thus,

$$|\mathbf{a}\times\mathbf{b}|=|a|\,|b|\,\sin\theta.\tag{2.33}$$

Clearly, $\mathbf{a}\times\mathbf{a}=\mathbf{0}$ for any vector, since θ is always zero in this case. Also, if $\mathbf{a}\times\mathbf{b}=\mathbf{0}$ then either $|a|=0$, $|b|=0$, or $\mathbf{b}$ is parallel (or antiparallel) to $\mathbf{a}$.

Consider the parallelogram defined by vectors $\mathbf{a}$ and $\mathbf{b}$ (see Fig. 6). The scalar area is $a\,b\sin\theta$. By definition, the vector area has the magnitude of the scalar area, and is normal to the plane of the parallelogram, which means that it is perpendicular to both $\mathbf{a}$ and $\mathbf{b}$. Clearly, the vector area is given by

$$\mathbf{S}=\mathbf{a}\times\mathbf{b},\tag{2.34}$$

with the sense obtained from the right-hand grip rule by rotating $\mathbf{a}$ on to $\mathbf{b}$.

Suppose that a force $\mathbf{F}$ is applied at position $\mathbf{r}$ (see Fig. 7). The moment, or torque, about the origin O is the product of the magnitude of the force and the length of the lever arm OQ. Thus, the magnitude of the moment is $|F|\,|r|\,\sin\theta$. The direction of the moment is conventionally the direction of the axis through O about which the force tries to rotate objects, in the sense determined by the

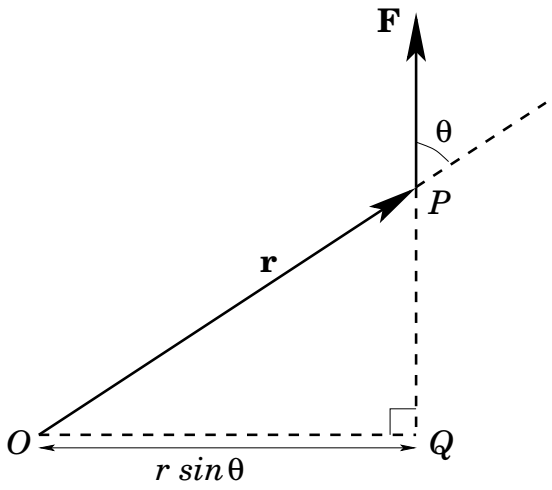

Figure 7:

right-hand grip rule. It follows that the vector moment is given by

$$\mathbf{M} = \mathbf{r} \times \mathbf{F}. \tag{2.35}$$

The angular momentum, $\mathbf{l}$, of a particle of linear momentum $\mathbf{p}$ and position vector $\mathbf{r}$ about the origin is simply defined as the moment of its momentum about the origin. Hence,

$$\mathbf{l} = \mathbf{r} \times \mathbf{p}. \tag{2.36}$$

2.5 Rotation

Let us try to define a rotation vector $\boldsymbol{\theta}$ whose magnitude is the angle of the rotation, θ, and whose direction is the axis of the rotation, in the sense determined by the right-hand grip rule. Is this a good vector? The short answer is, no. The problem is that the addition of rotations is not commutative, whereas vector addition is commuative. Figure 8 shows the effect of applying two successive 90° rotations, one about x-axis, and the other about the z-axis, to a six-sided die. In the left-hand case, the z-rotation is applied before the x-rotation, and *vice versa* in the right-hand case. It can be seen that the die ends up in two completely different states. Clearly, the z-rotation plus the x-rotation does not equal the x-rotation plus

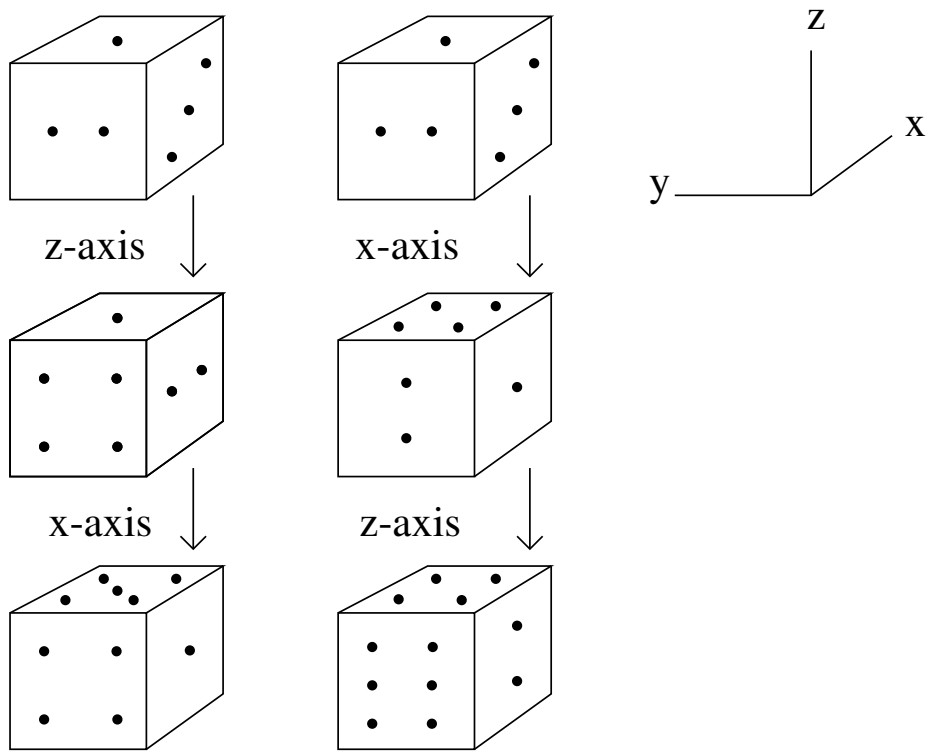

Figure 8:

the z-rotation. This non-commuting algebra cannot be represented by vectors. So, although rotations have a well-defined magnitude and direction, they are *not* vector quantities.

But, this is not quite the end of the story. Suppose that we take a general vector $\mathbf{a}$ and rotate it about the z-axis by a *small* angle $\delta\theta_z$. This is equivalent to rotating the basis about the z-axis by $-\delta\theta_z$. According to Eqs. (2.10)–(2.12), we have

$$\mathbf{a}' \simeq \mathbf{a} + \delta\theta_z\, \mathbf{e}_z \times \mathbf{a}, \tag{2.37}$$

where use has been made of the small angle expansions $\sin\theta \simeq \theta$ and $\cos\theta \simeq 1$. The above equation can easily be generalized to allow small rotations about the x- and y-axes by $\delta\theta_x$ and $\delta\theta_y$, respectively. We find that

$$\mathbf{a}' \simeq \mathbf{a} + \delta\boldsymbol{\theta} \times \mathbf{a}, \tag{2.38}$$

where

$$\delta\boldsymbol{\theta} = \delta\theta_x\, \mathbf{e}_x + \delta\theta_y\, \mathbf{e}_y + \delta\theta_z\, \mathbf{e}_z. \tag{2.39}$$

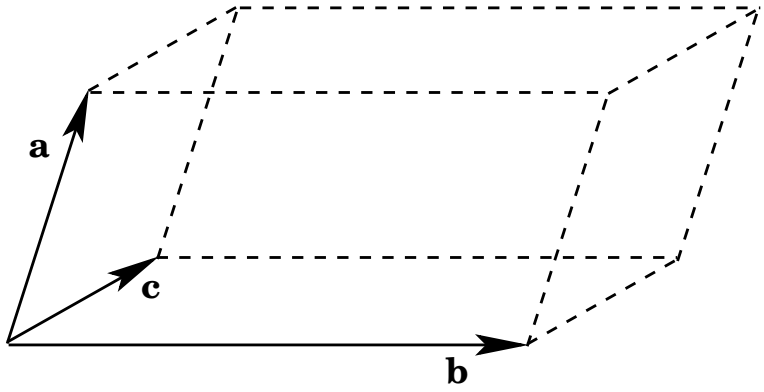

Figure 9:

Clearly, we can define a rotation vector $\delta\boldsymbol{\theta}$, but it only works for *small* angle rotations (*i.e.*, sufficiently small that the small angle expansions of sine and cosine are good). According to the above equation, a small z-rotation plus a small x-rotation is (approximately) equal to the two rotations applied in the opposite order. The fact that infinitesimal rotation is a vector implies that angular velocity,

$$\boldsymbol{\omega} = \lim_{\delta t \rightarrow 0} \frac{\delta\boldsymbol{\theta}}{\delta t}, \tag{2.40}$$

must be a vector as well. Also, if $\mathbf{a}'$ is interpreted as $\mathbf{a}(t+\delta t)$ in the above equation then it is clear that the equation of motion of a vector precessing about the origin with angular velocity $\boldsymbol{\omega}$ is

$$\frac{d\mathbf{a}}{dt} = \boldsymbol{\omega} \times \mathbf{a}. \tag{2.41}$$

2.6 The scalar triple product

Consider three vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$. The scalar triple product is defined $\mathbf{a}\cdot\mathbf{b}\times\mathbf{c}$. Now, $\mathbf{b}\times\mathbf{c}$ is the vector area of the parallelogram defined by $\mathbf{b}$ and $\mathbf{c}$. So, $\mathbf{a}\cdot\mathbf{b}\times\mathbf{c}$ is the scalar area of this parallelogram times the component of $\mathbf{a}$ in the direction of its normal. It follows that $\mathbf{a}\cdot\mathbf{b}\times\mathbf{c}$ is the *volume* of the parallelepiped defined by vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ (see Fig. 9). This volume is independent of how the triple product is formed from $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$, except that

$$\mathbf{a}\cdot\mathbf{b}\times\mathbf{c} = -\mathbf{a}\cdot\mathbf{c}\times\mathbf{b}. \tag{2.42}$$

So, the "volume" is positive if $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ form a right-handed set (*i.e.*, if $\mathbf{a}$ lies above the plane of $\mathbf{b}$ and $\mathbf{c}$, in the sense determined from the right-hand grip rule by rotating $\mathbf{b}$ onto $\mathbf{c}$) and negative if they form a left-handed set. The triple product is unchanged if the dot and cross product operators are interchanged:

$$\mathbf{a} \cdot \mathbf{b} \times \mathbf{c} = \mathbf{a} \times \mathbf{b} \cdot \mathbf{c}. \tag{2.43}$$

The triple product is also invariant under any cyclic permutation of $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$,

$$\mathbf{a} \cdot \mathbf{b} \times \mathbf{c} = \mathbf{b} \cdot \mathbf{c} \times \mathbf{a} = \mathbf{c} \cdot \mathbf{a} \times \mathbf{b}, \tag{2.44}$$

but any anti-cyclic permutation causes it to change sign,

$$\mathbf{a} \cdot \mathbf{b} \times \mathbf{c} = -\mathbf{b} \cdot \mathbf{a} \times \mathbf{c}. \tag{2.45}$$

The scalar triple product is zero if any two of $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ are parallel, or if $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ are co-planar.

If $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ are non-coplanar, then any vector $\mathbf{r}$ can be written in terms of them:

$$\mathbf{r} = \alpha\,\mathbf{a} + \beta\,\mathbf{b} + \gamma\,\mathbf{c}. \tag{2.46}$$

Forming the dot product of this equation with $\mathbf{b} \times \mathbf{c}$, we then obtain

$$\mathbf{r} \cdot \mathbf{b} \times \mathbf{c} = \alpha\,\mathbf{a} \cdot \mathbf{b} \times \mathbf{c}, \tag{2.47}$$

so

$$\alpha = \frac{\mathbf{r} \cdot \mathbf{b} \times \mathbf{c}}{\mathbf{a} \cdot \mathbf{b} \times \mathbf{c}}. \tag{2.48}$$

Analogous expressions can be written for β and γ. The parameters α, β, and γ are uniquely determined provided $\mathbf{a} \cdot \mathbf{b} \times \mathbf{c} \neq 0$: *i.e.*, provided that the three basis vectors are not co-planar.

2.7 The vector triple product

For three vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$, the vector triple product is defined $\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$. The brackets are important because $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) \neq (\mathbf{a} \times \mathbf{b}) \times \mathbf{c}$. In fact, it can be demonstrated that

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) \equiv (\mathbf{a} \cdot \mathbf{c})\,\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\,\mathbf{c} \tag{2.49}$$

and

$$(\mathbf{a}\times\mathbf{b})\times\mathbf{c}\equiv(\mathbf{a}\cdot\mathbf{c})\,\mathbf{b}-(\mathbf{b}\cdot\mathbf{c})\,\mathbf{a}. \tag{2.50}$$

Let us try to prove the first of the above theorems. The left-hand side and the right-hand side are both proper vectors, so if we can prove this result in one particular coordinate system then it must be true in general. Let us take convenient axes such that the x-axis lies along $\mathbf{b}$, and $\mathbf{c}$ lies in the x-y plane. It follows that $\mathbf{b}=(b_x,\,0,\,0)$, $\mathbf{c}=(c_x,\,c_y,\,0)$, and $\mathbf{a}=(a_x,\,a_y,\,a_z)$. The vector $\mathbf{b}\times\mathbf{c}$ is directed along the z-axis: $\mathbf{b}\times\mathbf{c}=(0,\,0,\,b_x\,c_y)$. It follows that $\mathbf{a}\times(\mathbf{b}\times\mathbf{c})$ lies in the x-y plane: $\mathbf{a}\times(\mathbf{b}\times\mathbf{c})=(a_y\,b_x\,c_y,\,-a_x\,b_x\,c_y,\,0)$. This is the left-hand side of Eq. (2.49) in our convenient coordinates. To evaluate the right-hand side, we need $\mathbf{a}\cdot\mathbf{c}=a_x\,c_x+a_y\,c_y$ and $\mathbf{a}\cdot\mathbf{b}=a_x\,b_x$. It follows that the right-hand side is

$$\begin{aligned}\text{RHS} &= (\,[a_x\,c_x+a_y\,c_y]\,b_x,\,0,\,0)-(a_x\,b_x\,c_x,\,a_x\,b_x\,c_y,\,0)\\ &= (a_y\,c_y\,b_x,\,-a_x\,b_x\,c_y,\,0)=\text{LHS},\end{aligned} \tag{2.51}$$

which proves the theorem.

2.8 Vector calculus

Suppose that vector $\mathbf{a}$ varies with time, so that $\mathbf{a}=\mathbf{a}(t)$. The time derivative of the vector is defined

$$\frac{d\mathbf{a}}{dt}=\lim_{\delta t\rightarrow 0}\left[\frac{\mathbf{a}(t+\delta t)-\mathbf{a}(t)}{\delta t}\right]. \tag{2.52}$$

When written out in component form this becomes

$$\frac{d\mathbf{a}}{dt}=\left(\frac{da_x}{dt},\frac{da_y}{dt},\frac{da_z}{dt}\right). \tag{2.53}$$

Suppose that $\mathbf{a}$ is, in fact, the product of a scalar $\phi(t)$ and another vector $\mathbf{b}(t)$. What now is the time derivative of $\mathbf{a}$? We have

$$\frac{da_x}{dt}=\frac{d}{dt}(\phi\,b_x)=\frac{d\phi}{dt}\,b_x+\phi\,\frac{db_x}{dt}, \tag{2.54}$$

which implies that

$$\frac{d\mathbf{a}}{dt}=\frac{d\phi}{dt}\,\mathbf{b}+\phi\,\frac{d\mathbf{b}}{dt}. \tag{2.55}$$

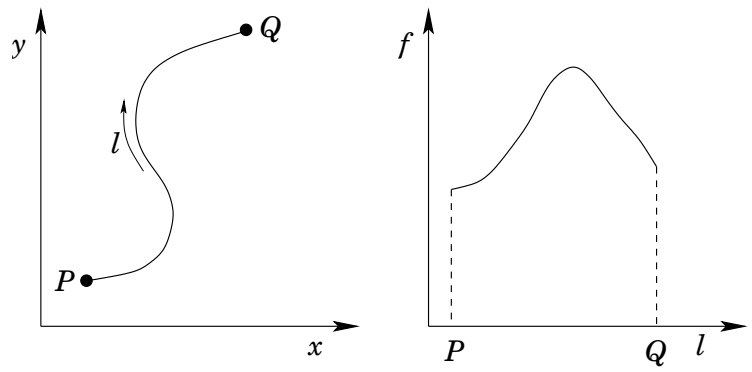

Figure 10:

It is easily demonstrated that

$$\frac{d}{dt}(\mathbf{a}\cdot\mathbf{b}) = \frac{d\mathbf{a}}{dt}\cdot\mathbf{b} + \mathbf{a}\cdot\frac{d\mathbf{b}}{dt}. \tag{2.56}$$

Likewise,

$$\frac{d}{dt}(\mathbf{a}\times\mathbf{b}) = \frac{d\mathbf{a}}{dt}\times\mathbf{b} + \mathbf{a}\times\frac{d\mathbf{b}}{dt}. \tag{2.57}$$

It can be seen that the laws of vector differentiation are analogous to those of conventional calculus.

2.9 Line integrals

Consider a two-dimensional function $f(x,y)$ which is defined for all x and y. What is meant by the integral of f along a given curve from P to Q in the x-y plane? We first draw out f as a function of length l along the path (see Fig. 10). The integral is then simply given by

$$\int_P^Q f(x,y)\,dl = \text{Area under the curve.} \tag{2.58}$$

As an example of this, consider the integral of $f(x,y) = x\,y$ between P and Q along the two routes indicated in Fig. 11. Along route 1 we have $x = y$, so $dl = \sqrt{2}\,dx$. Thus,

$$\int_P^Q x\,y\,dl = \int_0^1 x^2\,\sqrt{2}\,dx = \frac{\sqrt{2}}{3}. \tag{2.59}$$

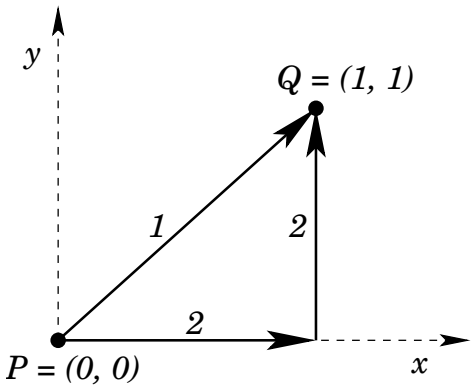

Figure 11:

The integration along route 2 gives

$$\begin{aligned}\int_P^Q x\,y\,dl &= \int_0^1 x\,y\,dx\Big|_{y=0} + \int_0^1 x\,y\,dy\Big|_{x=1} \\ &= 0 + \int_0^1 y\,dy = \frac{1}{2}. \end{aligned} \tag{2.60}$$

Note that the integral depends on the route taken between the initial and final points.

The most common type of line integral is that where the contributions from dx and dy are evaluated separately, rather that through the path length dl:

$$\int_P^Q \left[f(x,y)\,dx + g(x,y)\,dy\right]. \tag{2.61}$$

As an example of this, consider the integral

$$\int_P^Q \left[y^3\,dx + x\,dy\right] \tag{2.62}$$

along the two routes indicated in Fig. 12. Along route 1 we have $x = y + 1$ and $dx = dy$, so

$$\int_P^Q = \int_0^1 \left[y^3\,dy + (y+1)\,dy\right] = \frac{7}{4}. \tag{2.63}$$

Along route 2,

$$\int_P^Q = \int_1^2 y^3\,dx\Big|_{y=0} + \int_0^1 x\,dy\Big|_{x=2} = 2. \tag{2.64}$$

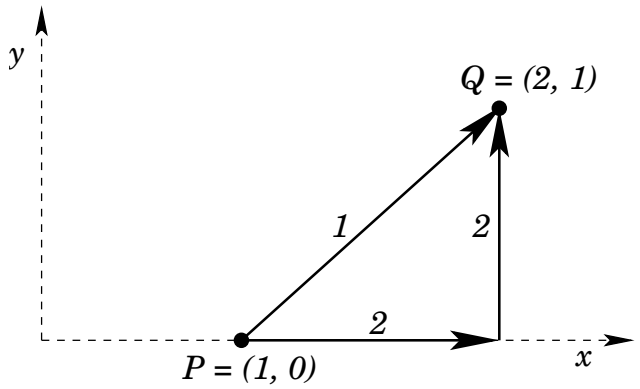

Figure 12:

Again, the integral depends on the path of integration.

Suppose that we have a line integral which does *not* depend on the path of integration. It follows that

$$\int_P^Q (f\,dx + g\,dy) = F(Q) - F(P) \tag{2.65}$$

for some function F. Given $F(P)$ for one point P in the x-y plane, then

$$F(Q) = F(P) + \int_P^Q (f\,dx + g\,dy) \tag{2.66}$$

defines $F(Q)$ for all other points in the plane. We can then draw a contour map of $F(x, y)$. The line integral between points P and Q is simply the change in height in the contour map between these two points:

$$\int_P^Q (f\,dx + g\,dy) = \int_P^Q dF(x, y) = F(Q) - F(P). \tag{2.67}$$

Thus,

$$dF(x, y) = f(x, y)\,dx + g(x, y)\,dy. \tag{2.68}$$

For instance, if $F = x\,y^3$ then $dF = y^3\,dx + 3\,x\,y^2\,dy$ and

$$\int_P^Q \left(y^3\,dx + 3\,x\,y^2\,dy\right) = \left[x\,y^3\right]_P^Q \tag{2.69}$$

is independent of the path of integration.

It is clear that there are two distinct types of line integral. Those which depend only on their endpoints and not on the path of integration, and those which depend both on their endpoints and the integration path.

2.10 Gradient

A one-dimensional function $f(x)$ has a gradient df/dx which is defined as the slope of the tangent to the curve at x. We wish to extend this idea to cover scalar fields in two and three dimensions.

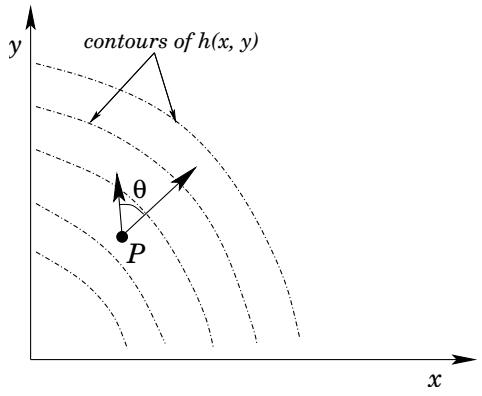

Figure 13:

Consider a two-dimensional scalar field $h(x, y)$, which is (say) the height of a hill. Let $d\mathbf{l} = (dx, dy)$ be an element of horizontal distance. Consider dh/dl, where dh is the change in height after moving an infinitesimal distance $d\mathbf{l}$. This quantity is somewhat like the one-dimensional gradient, except that dh depends on the *direction* of $d\mathbf{l}$, as well as its magnitude. In the immediate vicinity of some point P, the slope reduces to an inclined plane (see Fig. 13). The largest value of dh/dl is straight up the slope. For any other direction

$$\frac{dh}{dl} = \left(\frac{dh}{dl}\right)_{\max} \cos\theta. \tag{2.70}$$

Let us define a two-dimensional vector, $\mathbf{grad}\, h$, called the *gradient* of h, whose magnitude is $(dh/dl)_{\max}$, and whose direction is the direction up the steepest slope. Because of the $\cos\theta$ property, the component of $\mathbf{grad}\, h$ in any direction equals dh/dl for that direction.

The component of dh/dl in the x-direction can be obtained by plotting out the profile of h at constant y, and then finding the slope of the tangent to the curve at given x. This quantity is known as the *partial derivative* of h with respect to x at constant y, and is denoted $(\partial h/\partial x)_y$. Likewise, the gradient of the profile at

constant x is written $(\partial h/\partial y)_x$. Note that the subscripts denoting constant-x and constant-y are usually omitted, unless there is any ambiguity. If follows that in component form

$$\mathbf{grad}\, h = \left(\frac{\partial h}{\partial x}, \frac{\partial h}{\partial y}\right). \tag{2.71}$$

Now, the equation of the tangent plane at $P = (x_0, y_0)$ is

$$h_T(x, y) = h(x_0, y_0) + \alpha\,(x - x_0) + \beta\,(y - y_0). \tag{2.72}$$

This has the same local gradients as $h(x, y)$, so

$$\alpha = \frac{\partial h}{\partial x}, \qquad \beta = \frac{\partial h}{\partial y}, \tag{2.73}$$

by differentiation of the above. For small $dx = x - x_0$ and $dy = y - y_0$, the function h is coincident with the tangent plane. We have

$$dh = \frac{\partial h}{\partial x}\,dx + \frac{\partial h}{\partial y}\,dy, \tag{2.74}$$

but $\mathbf{grad}\, h = (\partial h/\partial x, \partial h/\partial y)$ and $d\mathbf{l} = (dx, dy)$, so

$$dh = \mathbf{grad}\, h \cdot d\mathbf{l}. \tag{2.75}$$

Incidentally, the above equation demonstrates that $\mathbf{grad}\, h$ is a proper vector, since the left-hand side is a scalar, and, according to the properties of the dot product, the right-hand side is also a scalar, provided that $d\mathbf{l}$ and $\mathbf{grad}\, h$ are both proper vectors ($d\mathbf{l}$ is an obvious vector, because it is directly derived from displacements).

Consider, now, a three-dimensional temperature distribution $T(x, y, z)$ in (say) a reaction vessel. Let us define $\mathbf{grad}\, T$, as before, as a vector whose magnitude is $(dT/dl)_{\max}$, and whose direction is the direction of the maximum gradient. This vector is written in component form

$$\mathbf{grad}\, T = \left(\frac{\partial T}{\partial x}, \frac{\partial T}{\partial y}, \frac{\partial T}{\partial z}\right). \tag{2.76}$$

Here, $\partial T/\partial x \equiv (\partial T/\partial x)_{y,z}$ is the gradient of the one-dimensional temperature profile at constant y and z. The change in T in going from point P to a neighbouring

point offset by $d\mathbf{l} = (dx,\, dy,\, dz)$ is

$$dT = \frac{\partial T}{\partial x}\, dx + \frac{\partial T}{\partial y}\, dy + \frac{\partial T}{\partial z}\, dz. \tag{2.77}$$

In vector form, this becomes

$$dT = \mathbf{grad}\, T \cdot d\mathbf{l}. \tag{2.78}$$

Suppose that $dT = 0$ for some $d\mathbf{l}$. It follows that

$$dT = \mathbf{grad}\, T \cdot d\mathbf{l} = 0. \tag{2.79}$$

So, $d\mathbf{l}$ is perpendicular to $\mathbf{grad}\, T$. Since $dT = 0$ along so-called "isotherms" (*i.e.*, contours of the temperature), we conclude that the isotherms (contours) are everywhere perpendicular to $\mathbf{grad}\, T$ (see Fig. 14).

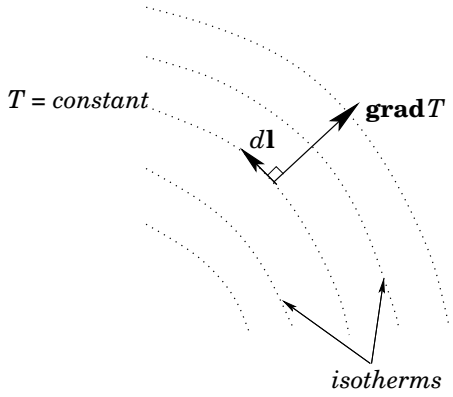

Figure 14:

It is, of course, possible to integrate dT. The line integral from point P to point Q is written

$$\int_P^Q dT = \int_P^Q \mathbf{grad}\, T \cdot d\mathbf{l} = T(Q) - T(P). \tag{2.80}$$

This integral is clearly independent of the path taken between P and Q, so $\int_P^Q \mathbf{grad}\, T \cdot d\mathbf{l}$ must be path independent.

In general, $\int_P^Q \mathbf{A} \cdot d\mathbf{l}$ depends on path, but for some special vector fields the integral is path independent. Such fields are called *conservative* fields. It can be

shown that if $\mathbf{A}$ is a conservative field then $\mathbf{A} = \mathbf{grad}\,\phi$ for some scalar field ϕ. The proof of this is straightforward. Keeping P fixed we have

$$\int_P^Q \mathbf{A}\cdot d\mathbf{l} = V(Q), \tag{2.81}$$

where $V(Q)$ is a well-defined function, due to the path independent nature of the line integral. Consider moving the position of the end point by an infinitesimal amount dx in the x-direction. We have

$$V(Q+dx) = V(Q) + \int_Q^{Q+dx} \mathbf{A}\cdot d\mathbf{l} = V(Q) + A_x\,dx. \tag{2.82}$$

Hence,

$$\frac{\partial V}{\partial x} = A_x, \tag{2.83}$$

with analogous relations for the other components of $\mathbf{A}$. It follows that

$$\mathbf{A} = \mathbf{grad}\,V. \tag{2.84}$$

In physics, the force due to gravity is a good example of a conservative field. If $\mathbf{A}$ is a force, then $\int \mathbf{A}\cdot d\mathbf{l}$ is the work done in traversing some path. If $\mathbf{A}$ is conservative then

$$\oint \mathbf{A}\cdot d\mathbf{l} = 0, \tag{2.85}$$

where $\oint$ corresponds to the line integral around some closed loop. The fact that zero net work is done in going around a closed loop is equivalent to the conservation of energy (this is why conservative fields are called "conservative"). A good example of a non-conservative field is the force due to friction. Clearly, a frictional system loses energy in going around a closed cycle, so $\oint \mathbf{A}\cdot d\mathbf{l} \neq 0$.

It is useful to define the vector *operator*

$$\nabla \equiv \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right), \tag{2.86}$$

which is usually called the *grad* or *del* operator. This operator acts on everything to its right in an expression, until the end of the expression or a closing bracket is reached. For instance,

$$\mathbf{grad}\,f = \nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right). \tag{2.87}$$

For two scalar fields ϕ and ψ,

$$\mathbf{grad}\,(\phi\,\psi) = \phi\;\mathbf{grad}\,\psi + \psi\;\mathbf{grad}\,\phi, \tag{2.88}$$

which can be written more succinctly as

$$\nabla(\phi\,\psi) = \phi\,\nabla\psi + \psi\,\nabla\phi. \tag{2.89}$$

Suppose that we rotate the basis about the z-axis by θ degrees. By analogy with Eqs. (2.7)–(2.9), the old coordinates $(x,\ y,\ z)$ are related to the new ones $(x',\ y',\ z')$ via

$$x = x'\,\cos\theta - y'\,\sin\theta, \tag{2.90}$$

$$y = x'\,\sin\theta + y'\,\cos\theta, \tag{2.91}$$

$$z = z'. \tag{2.92}$$

Now,

$$\frac{\partial}{\partial x'} = \left(\frac{\partial x}{\partial x'}\right)_{y',z'}\frac{\partial}{\partial x} + \left(\frac{\partial y}{\partial x'}\right)_{y',z'}\frac{\partial}{\partial y} + \left(\frac{\partial z}{\partial x'}\right)_{y',z'}\frac{\partial}{\partial z}, \tag{2.93}$$

giving

$$\frac{\partial}{\partial x'} = \cos\theta\,\frac{\partial}{\partial x} + \sin\theta\,\frac{\partial}{\partial y}, \tag{2.94}$$

and

$$\nabla_{x'} = \cos\theta\,\nabla_x + \sin\theta\,\nabla_y. \tag{2.95}$$

It can be seen that the differential operator ∇ transforms like a proper vector, according to Eqs. (2.10)–(2.12). This is another proof that ∇f is a good vector.

2.11 Useful vector formulae

Vector addition:

$$\mathbf{a} + \mathbf{b} \equiv (a_x + b_x,\ a_y + b_y,\ a_z + b_z)$$

Vector multiplication:

$$n\,\mathbf{a} \equiv (n\,a_x, n\,a_y, n\,a_z)$$

Scalar product:

$$\mathbf{a} \cdot \mathbf{b} = a_x\, b_x + a_y\, b_y + a_z\, b_z$$

Vector product:

$$\mathbf{a} \times \mathbf{b} = (a_y\, b_z - a_z\, b_y,\; a_z\, b_x - a_x\, b_z,\; a_x\, b_y - a_y\, b_x)$$

Scalar triple product:

$$\mathbf{a} \cdot \mathbf{b} \times \mathbf{c} = \mathbf{a} \times \mathbf{b} \cdot \mathbf{c} = \mathbf{b} \cdot \mathbf{c} \times \mathbf{a} = -\mathbf{b} \cdot \mathbf{a} \times \mathbf{c}$$

Vector triple product:

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\, \mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\, \mathbf{c}$$
$$(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} = (\mathbf{a} \cdot \mathbf{c})\, \mathbf{b} - (\mathbf{b} \cdot \mathbf{c})\, \mathbf{a}$$

2.12 Useful trigonometric formulae

$$\begin{aligned}
\cos^2 a + \sin^2 a &= 1, \\
\sin(a+b) &= \sin a\, \cos b + \cos a\, \sin b, \\
\cos(a+b) &= \cos a\, \cos b - \sin a\, \sin b, \\
\tan(a+b) &= \frac{\tan a + \tan b}{1 - \tan a\, \tan b}, \\
\sin(2\,a) &= 2\, \sin a\, \cos a, \\
\cos(2\,a) &= 2\, \cos^2 a - 1 = 1 - 2\, \sin^2 a, \\
2\, \sin a\, \sin b &= \cos(a-b) - \cos(a+b), \\
2\, \cos a\, \cos b &= \cos(a-b) + \cos(a+b), \\
2\, \sin a\, \cos b &= \sin(a-b) + \sin(a+b), \\
\sin a + \sin b &= 2\, \sin\left(\frac{a+b}{2}\right)\, \cos\left(\frac{a-b}{2}\right), \\
\sin a - \sin b &= 2\, \cos\left(\frac{a+b}{2}\right)\, \sin\left(\frac{a-b}{2}\right),
\end{aligned}$$

$$
\begin{aligned}
\cos\, a + \cos\, b &= 2\, \cos\left(\frac{a+b}{2}\right)\, \cos\left(\frac{a-b}{2}\right), \\
\cos\, a - \cos\, b &= -2\, \sin\left(\frac{a+b}{2}\right)\, \sin\left(\frac{a-b}{2}\right), \\
\sin^{-1}(a) &= \int_0^a \frac{dy}{(1-y^2)^{1/2}}, \\
\tan^{-1}(a) &= \int_0^a \frac{dy}{1+y^2}, \\
\cosh^2 a - \sinh^2 a &= 1, \\
\sinh\,(a+b) &= \sinh a\, \cosh b + \cosh a\, \sinh b, \\
\cosh\,(a+b) &= \cosh a\, \cosh b + \sinh a\, \sinh b, \\
\tanh\,(a+b) &= \frac{\tanh a + \tanh b}{1 + \tanh a\, \tanh b}, \\
\sinh\,(2\,a) &= 2\, \sinh a\, \cosh a, \\
\cosh\,(2\,a) &= 2\, \cosh^2 a - 1 = 2\, \sinh^2 a + 1, \\
2\, \sinh a\, \sinh b &= \cosh\,(a+b) - \cosh\,(a-b), \\
2\, \cosh a\, \cosh b &= \cosh\,(a+b) + \cos\,(a-b), \\
2\, \sinh a\, \cosh b &= \sinh\,(a+b) + \sinh\,(a-b), \\
\sinh\, a + \sinh\, b &= 2\, \sinh\left(\frac{a+b}{2}\right)\, \cosh\left(\frac{a-b}{2}\right), \\
\sinh\, a - \sinh\, b &= 2\, \cosh\left(\frac{a+b}{2}\right)\, \sinh\left(\frac{a-b}{2}\right), \\
\cosh\, a + \cosh\, b &= 2\, \cosh\left(\frac{a+b}{2}\right)\, \cosh\left(\frac{a-b}{2}\right), \\
\cosh\, a - \cosh\, b &= 2\, \sinh\left(\frac{a+b}{2}\right)\, \sinh\left(\frac{a-b}{2}\right), \\
\sinh^{-1}(a) &= \int_0^a \frac{dy}{(1+y^2)^{1/2}}, \\
\tanh^{-1}(a) &= \int_0^a \frac{dy}{1-y^2}.
\end{aligned}
$$

3 Fundamentals

3.1 Introduction

In this section, we shall examine the fundamental concepts which underlie all of classical dynamics.

3.2 Fundamental assumptions

Classical dynamics is a mathematical model which aims to both describe and predict the motions of the various objects which we encounter in the world around us. The general principles of this theory were first enunciated by Sir Isaac Newton in a work entitled *Philosophiae Naturalis Principia Mathematica* (1687), which is commonly known as the *Principa*.

Up until the beginning of the 20th century, Newton's theory of motion was thought to constitute a *complete* description of all types of motion occurring in the Universe. We now know that this is not the case. The modern view is that Newton's theory is an *approximation* which is generally valid when describing the *low speed* (compared to the speed of light) motions of *macroscopic* objects. Newton's theory breaks down, and must be replaced by Einstein's theory of relativity, when objects start to move at speeds approaching the speed of light. Newton's theory also breaks down on the atomic scale, and must be replaced by quantum mechanics.

Newton's theory of motion is an *axiomatic system*. Like all axiomatic systems (*e.g.*, Euclidean geometry), it starts from a set of terms which are *undefined* within the theory. In the present case, the fundamental terms are *mass*, *position*, *time*, and *force*. It is taken for granted that we understand what these terms mean, and, in fact, that they correspond to *measurable* quantities which can be ascribed to, or associated with, objects in the world around us. The next component of an axiomatic system is a set of *axioms*. These are a set of *unproven* propositions, involving the undefined terms, from which all other propositions in the system can be derived via logic and mathematical analysis. In the present case, the axioms

are called *Newton's laws of motion*, and can only be justified via experimental observation.

In the following, it is assumed that we know how to set up a *Cartesian frame of reference*, and also know how to measure the positions of objects as functions of time within that frame. In addition, it is assumed that we have some familiarity with the laws of mechanics, and that we understand standard mathematics up to, and including, calculus, as well as the vector analysis outlined in Sect. 2.

3.3 Newton's first law of motion

As is well-known, Newton's first law of motion states that an object subject to zero net external force moves in a straight-line with constant speed (*i.e.*, it does not accelerate). However, this is only true in special frames of reference called *inertial frames*. Indeed, we can think of Newton's first law as the *definition* of an inertial frame: *i.e.*, an inertial frame of reference is one in which an object subject to zero net external force moves in a straight-line with constant speed.

Suppose that we have found an inertial frame of reference. Let us set up a Cartesian coordinate system in this frame. The motion of a point object can now be specified by giving its position vector, $\mathbf{r} = (x, y, z)$, with respect to the origin of our coordinate system, as a function of time, t. Consider a second frame of reference moving with some *constant* velocity $\mathbf{u}$ with respect to our first frame. Without loss of generality, we can suppose that the Cartesian axes in the second frame are parallel to the corresponding axes in the first frame, and that $\mathbf{u} = (u, 0, 0)$, and, finally, that the origins of the two frames instantaneously coincide at $t = 0$ (see Fig. 15). Suppose that the position vector of our point object is $\mathbf{r}' = (x', y', z')$ in the second frame of reference. It is fairly obvious, from Fig. 15, that at any given time, t, the coordinates of the object in the two reference frames satisfy

$$x' = x - u\,t, \tag{3.1}$$

$$y' = y, \tag{3.2}$$

$$z' = z. \tag{3.3}$$

This transformation law is generally known as the *Galilean transformation*, after

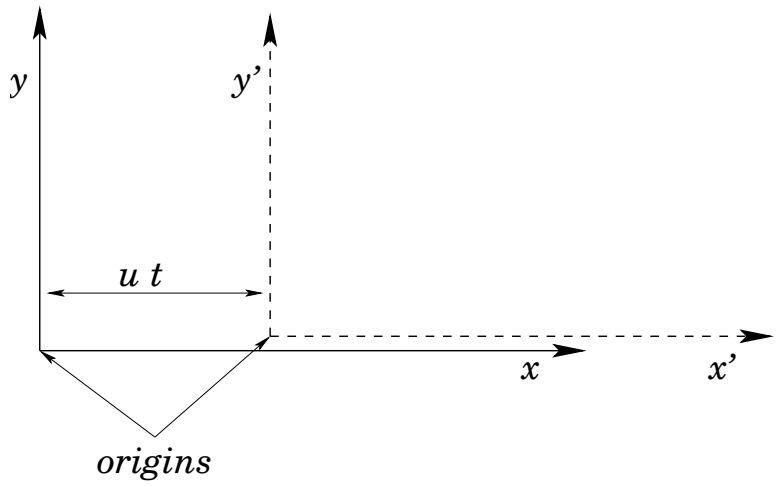

Figure 15:

Galileo.

The instantaneous velocity of the object in our first reference frame is given by $\mathbf{v} = d\mathbf{r}/dt = (dx/dt, dy/dt, dz/dt)$, with an analogous expression for the velocity, $\mathbf{v}'$, in the second frame. It follows from Eqs. (3.1)–(3.3) that the velocity components in the two frames satisfy

$$v'_x = v_x - u, \tag{3.4}$$

$$v'_y = v_y, \tag{3.5}$$

$$v'_z = v_z. \tag{3.6}$$

These equations can be written more succinctly as

$$\mathbf{v}' = \mathbf{v} - \mathbf{u}. \tag{3.7}$$

Finally, the instantaneous acceleration of the object in our first reference frame is given by $\mathbf{a} = d\mathbf{v}/dt = (dv_x/dt, dv_y/dt, dv_z/dt)$, with an analogous expression for the acceleration, $\mathbf{a}'$, in the second frame. It follows from Eqs. (3.4)–(3.6) that the acceleration components in the two frames satisfy

$$a'_x = a_x, \tag{3.8}$$

$$a'_y = a_y, \tag{3.9}$$

$$a'_z = a_z. \tag{3.10}$$

These equations can be written more succinctly as

$$\mathbf{a}' = \mathbf{a}. \tag{3.11}$$

According to Eqs. (3.7) and (3.11), if an object is moving in a straight-line with constant speed in our original inertial frame (*i.e.*, if $\mathbf{a} = \mathbf{0}$) then it also moves in a (different) straight-line with (a different) constant speed in the second frame of reference (*i.e.*, $\mathbf{a}' = \mathbf{0}$). Hence, we conclude that the second frame of reference is *also* an inertial frame.

A simple extension of the above argument allows us to conclude that there are an *infinite* number of different inertial frames moving with *constant velocities* with respect to one another.

But, what happens if the second frame of reference *accelerates* with respect to the first? In this case, it is not hard to see that Eq. (3.11) generalizes to

$$\mathbf{a}' = \mathbf{a} - \frac{d\mathbf{u}}{dt}, \tag{3.12}$$

where $\mathbf{u}(t)$ is the instantaneous velocity of the second frame with respect to the first. According to the above formula, if an object is moving in a straight-line with constant speed in the first frame (*i.e.*, if $\mathbf{a} = \mathbf{0}$) then it does not move in a straight-line with constant speed in the second frame (*i.e.*, $\mathbf{a}' \neq \mathbf{0}$). Hence, if the first frame is an inertial frame then the second is *not*.

A simple extension of the above argument allows us to conclude that any frame of reference which accelerates with respect to any inertial frame is not an inertial frame.

3.4 Newton's second law of motion

As is well-known, Newton's second law of motion states that if an object is subject to an external force, $\mathbf{f}$, then its equation of motion is given by

$$\frac{d\mathbf{p}}{dt} = \mathbf{f}, \tag{3.13}$$

where the momentum, $\mathbf{p}$, is the product of the object's inertial mass, m, and its velocity, $\mathbf{v}$. If m is not a function of time then the above expression reduces to the familiar equation

$$m\,\frac{d\mathbf{v}}{dt} = \mathbf{f}. \tag{3.14}$$

Note that this equation is only valid in a *inertial frame*. Clearly, the inertial mass of an object measures its reluctance to deviate from its preferred state of uniform motion in a straight-line (in an inertial frame). Of course, the above equation of motion can only be solved if we have an independent expression for the force, $\mathbf{f}$. Let us suppose that this is the case.

An important corollary of Newton's second law is that force is a *vector quantity*. This must be the case, since the law equates force to the product of a scalar (mass) and a vector (acceleration). Note that acceleration is obviously a vector because it is directly related to displacement, which is the prototype of all vectors (see Sect. 2). One consequence of force being a vector is that two forces, $\mathbf{f}_1$ and $\mathbf{f}_2$, both acting at a given point, have the same effect as a single force, $\mathbf{f} = \mathbf{f}_1 + \mathbf{f}_2$, acting at the same point, where the summation is performed according to the laws of vector addition (see Sect. 2). Likewise, a single force, $\mathbf{f}$, acting at a given point, has the same effect as two forces, $\mathbf{f}_1$ and $\mathbf{f}_2$, acting at the same point, provided that $\mathbf{f}_1 + \mathbf{f}_2 = \mathbf{f}$. This method of combining and splitting forces is known as the *resolution of forces*, and lies at the heart of many calculations in classical dynamics.

Taking the scalar product of Eq. (3.14) with the velocity, $\mathbf{v}$, we obtain

$$m\,\mathbf{v}\cdot\frac{d\mathbf{v}}{dt} = \frac{m}{2}\frac{d(\mathbf{v}\cdot\mathbf{v})}{dt} = \frac{m}{2}\frac{dv^2}{dt} = \mathbf{f}\cdot\mathbf{v}. \tag{3.15}$$

This can be written

$$\frac{dK}{dt} = \mathbf{f}\cdot\mathbf{v}. \tag{3.16}$$

where

$$K = \frac{1}{2}\,m\,v^2. \tag{3.17}$$

The right-hand side of Eq. (3.16) represents the rate at which the force does work on the object: *i.e.*, the rate at which the force transfers energy to the object. The quantity K represents the energy the object possesses by virtue of its motion. This

type of energy is generally known as *kinetic energy*. Thus, Eq. (3.16) states that any work done on an object by an external force goes to increase the object's kinetic energy.

Suppose that, under the action of the force, $\mathbf{f}$, our object moves from point P at time t_1 to point Q at time t_2. The net change in the object's kinetic energy is obtained by integrating Eq. (3.16):

$$\Delta K = \int_{t_1}^{t_2} \mathbf{f}\cdot\mathbf{v}\,dt = \int_P^Q \mathbf{f}\cdot d\mathbf{r}, \tag{3.18}$$

since $\mathbf{v} = d\mathbf{r}/dt$. Here, $d\mathbf{r}$ is an element of the object's path between points P and Q.

As described in Sect. 2.10, there are basically two kinds of forces in nature. Firstly, those for which line integrals of the type $\int_P^Q \mathbf{f}\cdot d\mathbf{r}$ depend on the end points, but not on the path taken between these points. Secondly, those for which line integrals of the type $\int_P^Q \mathbf{f}\cdot d\mathbf{r}$ depend both on the end points, and the path taken between these points. The first kind of force is termed *conservative*, whereas the second kind is termed *non-conservative*. It was also demonstrated in Sect. 2.10 that if the line integral $\int_P^Q \mathbf{f}\cdot d\mathbf{r}$ is *path-independent* then the force $\mathbf{f}$ can always be written as the gradient of a scalar potential. In other words, all conservative forces satisfy

$$\mathbf{f} = -\nabla U, \tag{3.19}$$

for some scalar potential $U(\mathbf{r})$. Note that

$$\int_P^Q \nabla U\cdot d\mathbf{r} = \Delta U = U(Q) - U(P), \tag{3.20}$$

irrespective of the path taken between P and Q. Hence. it follows from Eq. (3.18) that

$$\Delta K = -\Delta U \tag{3.21}$$

for conservative forces. Another way of writing this is

$$E = K + U = \text{constant}. \tag{3.22}$$

Of course, we recognize this as an *energy conservation equation*: E is the object's total energy, which is conserved; K is the energy the object has by virtue of its

motion, otherwise know as its *kinetic energy*; and U is the energy the object has by virtue of its position, otherwise known as its *potential energy*. Note, however, that we can only write such energy conservation equations for conservative forces (hence, the name). Gravity is a good example of a conservative force. Non-conservative forces, on the other hand, do not conserve energy. In general, this is because of some sort of frictional energy loss which drains energy from the dynamical system whilst it remains in motion. Note that potential energy is undefined to an arbitrary additive constant. In fact, it is only the *difference* in potential energy between different points in space which is well-defined.

3.5 Newton's third law of motion

Consider a system of N mutually interacting point mass objects. Let the ith object, whose mass is m_i, be located at vector displacement $\mathbf{r}_i$. Suppose that this object exerts a force $\mathbf{f}_{ji}$ on the jth object. Likewise, suppose that the jth object exerts a force f_{ij} on the ith object. As is well-known, Newton's third law of motion states that these two forces are equal and opposite, irrespective of their nature. In other words,

$$\mathbf{f}_{ij} = -\mathbf{f}_{ji}. \tag{3.23}$$

One corollary of Newton's third law is that an object cannot exert a force on itself.

In an inertial frame, Newton's second law of motion applied to the ith object yields

$$m_i \frac{d^2\mathbf{r}_i}{dt^2} = \sum_{j=1,N}^{j\neq i} \mathbf{f}_{ij}. \tag{3.24}$$

Note that the summation on the right-hand side of the above equation excludes the case $j = i$, since the ith object cannot exert a force on itself. Let us now take the above equation and sum it over all objects. We obtain

$$\sum_{i=1,N} m_i \frac{d^2\mathbf{r}_i}{dt^2} = \sum_{i,j=1,N}^{j\neq i} \mathbf{f}_{ij}. \tag{3.25}$$

Consider the sum over forces on the right-hand side of the above equation. Each element of this sum—$\mathbf{f}_{ij}$, say—can be paired with another element—$\mathbf{f}_{ji}$, in this

case—which is equal and opposite. In other words, the elements of the sum all cancel out in pairs. Thus, the net value of the sum is *zero*. It follows that the above equation can be written

$$M\,\frac{d^2\mathbf{r}_{cm}}{dt^2} = \mathbf{0}, \tag{3.26}$$

where $M = \sum_{i=1}^N m_i$ is the total mass. The quantity $\mathbf{r}_{cm}$ is the vector displacement of the *center of mass* of the system, which is an imaginary point whose coordinates are the mass weighted averages of the coordinates of the objects which constitute the system. Thus,

$$\mathbf{r}_{cm} = \frac{\sum_{i=1}^N m_i\,\mathbf{r}_i}{\sum_{i=1}^N m_i}. \tag{3.27}$$

According to Eq. (3.26), the center of mass of the system moves in a uniform straight-line, in accordance with Newton's first law of motion, irrespective of the nature of the forces acting between the various components of the system.

Now, if the center of mass moves in a uniform straight-line, then the center of mass velocity,

$$\frac{d\mathbf{r}_{cm}}{dt} = \frac{\sum_{i=1}^N m_i\,d\mathbf{r}_i/dt}{\sum_{i=1}^N m_i}, \tag{3.28}$$

is a constant of the motion. However, the momentum of the ith object takes the form $\mathbf{p}_i = m_i\,d\mathbf{r}_i/dt$. Hence, the total momentum of the system is written

$$\mathbf{P} = \sum_{i=1}^N m_i\,\frac{d\mathbf{r}_i}{dt}. \tag{3.29}$$

A comparison of Eqs. (3.28) and (3.29) suggests that $\mathbf{P}$ is also a constant of the motion. In other words, the total momentum of the system is a *conserved* quantity, irrespective of the nature of the forces acting between the various components of the system. This result is a direct consequence of Newton's third law of motion.

Taking the vector product Eq. (3.24) with the position vector $\mathbf{r}_i$, we obtain

$$m_i\,\mathbf{r}_i\times\frac{d^2\mathbf{r}_i}{dt^2} = \sum_{j=1,N}^{j\neq i}\mathbf{r}_i\times\mathbf{f}_{ij}. \tag{3.30}$$

However, it is easily seen that

$$m_i\,\mathbf{r}_i\times\frac{d^2\mathbf{r}_i}{dt^2}=\frac{d(m_i\,\mathbf{r}_i\times d\mathbf{r}_i/dt)}{dt}=\frac{d\mathbf{l}_i}{dt},\tag{3.31}$$

where

$$\mathbf{l}_i=m_i\,\mathbf{r}_i\times\frac{d\mathbf{r}_i}{dt}\tag{3.32}$$

is the *angular momentum* of the ith particle about the origin of our coordinate system. The total angular momentum of the system (about the origin) takes the form

$$\mathbf{L}=\sum_{i=1,N}\mathbf{l}_i\tag{3.33}$$

Hence, summing Eq. (3.30) over all particles, we obtain

$$\frac{d\mathbf{L}}{dt}=\sum_{i,j=1,N}^{i\neq j}\mathbf{r}_i\times\mathbf{f}_{ij}.\tag{3.34}$$

Consider the sum on the right-hand side of the above equation. A general term, $\mathbf{r}_i\times\mathbf{f}_{ij}$, in this sum can always be paired with a matching term, $\mathbf{r}_j\times\mathbf{f}_{ji}$, in which the indices have been swapped. Making use of Eq. (3.23), the sum of a general matched pair can be written

$$\mathbf{r}_i\times\mathbf{f}_{ij}+\mathbf{r}_j\times\mathbf{f}_{ji}=(\mathbf{r}_i-\mathbf{r}_j)\times\mathbf{f}_{ij}.\tag{3.35}$$

Suppose, now, that the forces acting between the various components of the system are *central* in nature, so that $\mathbf{f}_{ij}$ is parallel to $\mathbf{r}_i-\mathbf{r}_j$. In other words, the force exerted on object j by object i either points directly towards, or directly away from, object i, and *vice versa*. This is not a particularly onerous constraint, since most forces in nature are of this type (*e.g.*, gravity). It follows that if the forces are central in nature then the vector product in the above expression is zero. We conclude that

$$\mathbf{r}_i\times\mathbf{f}_{ij}+\mathbf{r}_j\times\mathbf{f}_{ji}=\mathbf{0},\tag{3.36}$$

for all values of i and j. Thus, the sum on the right-hand side of Eq. (3.34) is zero for any kind of central force. We are left with

$$\frac{d\mathbf{L}}{dt}=\mathbf{0}.\tag{3.37}$$

In other words, the total angular momentum of the system is a *conserved* quantity, provided that the different components of the system interact via *central* forces.

4 One-dimensional motion

4.1 Introduction

In this section, we shall use Newton's laws of motion to investigate various aspects of one-dimensional motion. Particular attention will be given to the various mathematical techniques generally used to analyze oscillatory motion.

4.2 Motion in a general one-dimensional potential

Consider an object of mass m moving in the x-direction, say, under the action of some x-directed force $f(x)$. Suppose that f is a conservative force, such as gravity. In this case, according to Eq. (3.19), we can write

$$f(x) = -\frac{dU(x)}{dx}, \tag{4.1}$$

where $U(x)$ is the potential energy of the object at position x. It is generally most convenient to specify a conservative force, such as $f(x)$, in terms of its associated potential energy function, $U(x)$.

Suppose that the curve $U(x)$ in Fig. 16 represents the potential energy of some mass m moving in a one-dimensional conservative force-field. For instance, $U(x)$ might represent the gravitational potential energy of a cyclist freewheeling in a hilly region. Note that we have set the potential energy at infinity to zero. This is a useful, and quite common, convention (recall that potential energy is undefined to within an arbitrary additive constant). What can we deduce about the motion of the mass in this potential?

We know that the total energy, E—which is the sum of the kinetic energy, K, and the potential energy, U—is a *constant* of the motion—see Eq. (3.22). Hence, we can write

$$K(x) = E - U(x). \tag{4.2}$$

Now, we also know that a kinetic energy can never be negative [since $K = (1/2)\,m\,v^2$, and neither m nor v^2 can be negative], so the above expression tells us that the

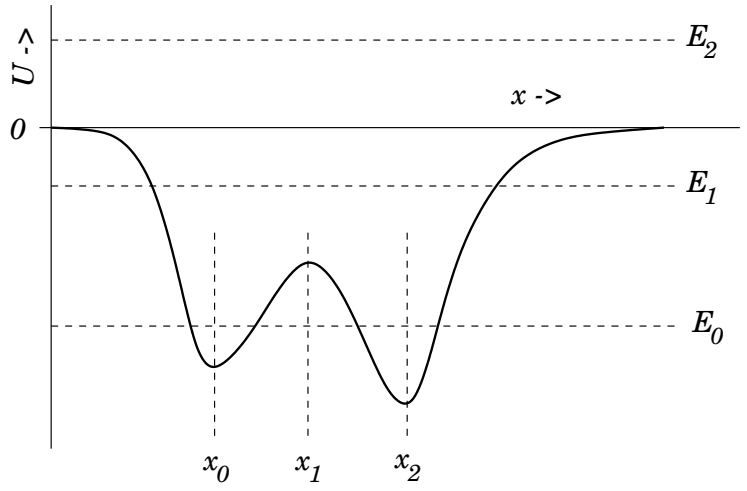

Figure 16:

motion of the mass is restricted to the region (or regions) in which the potential energy curve $U(x)$ falls below the value E. This idea is illustrated in Fig. 16. Suppose that the total energy of the system is E_0. It is clear, from the figure, that the mass is trapped inside one or other of the two dips in the potential—these dips are generally referred to as *potential wells*. Suppose that we now raise the energy to E_1. In this case, the mass is free to enter or leave each of the potential wells, but its motion is still *bounded* to some extent, since it clearly cannot move off to infinity. Finally, let us raise the energy to E_2. Now the mass is *unbounded*: *i.e.*, it can move off to infinity. In systems in which it makes sense to adopt the convention that the potential energy at infinity is zero, bounded systems are characterized by $E < 0$, whereas unbounded systems are characterized by $E > 0$.

The above discussion suggests that the motion of a mass moving in a potential generally becomes less bounded as the total energy E of the system increases. Conversely, we would expect the motion to become more bounded as E decreases. In fact, if the energy becomes sufficiently small, it appears likely that the system will settle down in some *equilibrium state* in which the mass is stationary. Let us try to identify any prospective equilibrium states in Fig. 16. If the mass remains stationary then it must be subject to zero force (otherwise it would accelerate). Hence, according to Eq. (4.1), an equilibrium state is characterized by

$$\frac{dU}{dx} = 0. \tag{4.3}$$

In other words, a equilibrium state corresponds to either a *maximum* or a *minimum* of the potential energy curve $U(x)$. It can be seen that the $U(x)$ curve shown in Fig. 16 has three associated equilibrium states: these are located at $x = x_0$, $x = x_1$, and $x = x_2$.

Let us now make a distinction between *stable equilibrium* points and *unstable equilibrium* points. When the system is slightly perturbed from a stable equilibrium point then the resultant force f should always be such as to attempt to return the system to this point. In other words, if $x = x_0$ is an equilibrium point, then we require

$$\left.\frac{df}{dx}\right|_{x_0} < 0 \tag{4.4}$$

for stability: *i.e.*, if the system is perturbed to the right, so that $x - x_0 > 0$, then the force must act to the left, so that $f < 0$, and *vice versa*. Likewise, if

$$\left.\frac{df}{dx}\right|_{x_0} > 0 \tag{4.5}$$

then the equilibrium point $x = x_0$ is unstable. It follows, from Eq. (4.1), that stable equilibrium points are characterized by

$$\frac{d^2U}{dx^2} > 0. \tag{4.6}$$

In other words, a stable equilibrium point corresponds to a *minimum* of the potential energy curve $U(x)$. Likewise, an unstable equilibrium point corresponds to a *maximum* of the $U(x)$ curve. Hence, we conclude that $x = x_0$ and $x = x_2$ are stable equilibrium points, in Fig. 16, whereas $x = x_1$ is an unstable equilibrium point. Of course, this makes perfect sense if we think of $U(x)$ as a gravitational potential energy curve, in which case U is directly proportional to height. All we are saying is that it is easy to confine a low energy mass at the bottom of a valley, but very difficult to balance the same mass on the top of a hill (since any slight perturbation to the mass will cause it to fall down the hill). Note, finally, that if

$$\frac{dU}{dx} = \frac{d^2U}{dx^2} = 0 \tag{4.7}$$

at any point (or in any region) then we have what is known as a *neutral equilibrium* point. We can move the mass slightly away from such a point and it will still

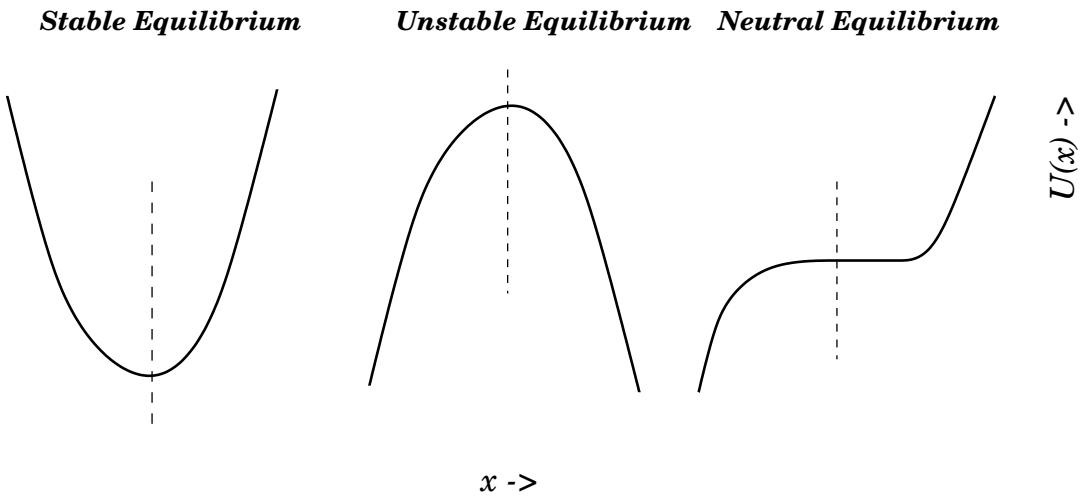

Figure 17:

remain in equilibrium (*i.e.*, it will neither attempt to return to its initial state, nor will it continue to move). A neutral equilibrium point corresponds to a *flat spot* in a $U(x)$ curve. See Fig. 17.

The equation of motion of an object moving in one dimension under the action of a conservative force is, in principle, integrable. Since $K = (1/2)\,m\,v^2$, the energy conservation equation (4.2) can be rearranged to give

$$v = \pm \left(\frac{2\,[E - U(x)]}{m}\right)^{1/2}, \tag{4.8}$$

where the $\pm$ signs correspond to motion to the left and to the right, respectively. However, given that $v = dx/dt$, this expression can be integrated, yielding

$$t = \pm \left(\frac{m}{2\,E}\right)^{1/2} \int_{x_0}^{x} \frac{dx'}{\sqrt{1 - U(x')/E}}, \tag{4.9}$$

where $x(t = 0) = x_0$. For sufficiently simple potential functions, $U(x)$, the above equation can be solved to give x as a function of t.

4.3 Velocity dependent forces

Consider an object of mass m moving in one dimension under the action of a force, f, which is a function of the object's speed, v, but not of its displacement,

x. Note that such a force is intrinsically non-conservative [since it clearly cannot be expressed as minus the gradient of some potential function, $U(x)$]. Now, the object's equation of motion is written

$$m\,\frac{dv}{dt} = f(v). \tag{4.10}$$

Integrating this equation, we obtain

$$\int_{v_0}^{v} \frac{dv'}{f(v')} = \frac{t}{m}, \tag{4.11}$$

where $v(t=0) = v_0$. In principle, the above equation can be solved to give $v(t)$. The equation of motion is also written

$$m\,v\,\frac{dv}{dx} = f(v), \tag{4.12}$$

since $v = dx/dt$. Integrating this equation, we obtain

$$\int_{v_0}^{v} \frac{v'\,dv'}{f(v')} = \frac{x - x_0}{m}, \tag{4.13}$$

where $x(t=0) = x_0$. In principle, the above equation can be solved to give $v(x)$.

Let us now consider a specific example. Suppose that an object of mass m falls vertically under gravity. Let x be the height through which the object has fallen since $t = 0$, at which time the object is assumed to be at rest. It follows that $x_0 = v_0 = 0$. Suppose that, in addition to the force of gravity, our object is subject to a retarding air resistance force which is proportional to the square of its instantaneous velocity. The object's equation of motion is thus

$$m\,\frac{dv}{dt} = m\,g - c\,v^2, \tag{4.14}$$

where $c > 0$. This equation can be integrated to give

$$\int_0^{v} \frac{dv'}{1 - (v/v_t)^2} = g\,t, \tag{4.15}$$

where $v_t = (m\,g/c)^{1/2}$. Making a change of variable, we obtain

$$\int_0^{v/v_t} \frac{dy}{1 - y^2} = \frac{g}{v_t}\,t. \tag{4.16}$$

The left-hand side of the above equation is now a standard integral, which can be solved to give

$$\tanh^{-1}\left(\frac{v}{v_t}\right) = \frac{g\,t}{v_t}, \tag{4.17}$$

or

$$v = v_t\,\tanh\left(\frac{g\,t}{v_t}\right). \tag{4.18}$$

Thus, when $t \ll v_t/g$, we obtain the standard result $v \simeq g\,t$, since $\tanh x \sim x$ for $x \ll 1$. However, when $t \gg v_t/g$, we get $v \simeq v_t$, since $\tanh x \simeq 1$ for $x \gg 1$. It follows that air resistance prevents the downward velocity of our object from increasing indefinitely as it falls. Instead, at large times, the velocity asymptotically approaches the so-called *terminal velocity*, v_t (at which the gravitational and air resistance forces balance).

The equation of motion of our falling object is also written

$$m\,v\,\frac{dv}{dx} = m\,g - c\,v^2. \tag{4.19}$$

This equation can be integrated to give

$$\int_0^v \frac{v'\,dv'}{1-(v/v_t)^2} = g\,x. \tag{4.20}$$

Making a change of variable, we obtain

$$\int_0^{(v/v_t)^2} \frac{dy}{1-y} = \frac{x}{x_t}, \tag{4.21}$$

where $x_t = m/(2\,c)$. The left-hand side of the above equation is now a standard integral, which can be solved to give

$$-\ln\left[1-\left(\frac{v}{v_t}\right)^2\right] = \frac{x}{x_t}, \tag{4.22}$$

or

$$v = v_t\left(1-\mathrm{e}^{-x/x_t}\right). \tag{4.23}$$

It follows that our object needs to fall a distance of order x_t before it achieves its terminal velocity.

4.4 Simple harmonic motion

Consider the motion of an object of mass m which is slightly perturbed from a stable equilibrium point at $x = 0$. Suppose that the object is moving in the conservative force-field $f(x)$. According to the analysis in the previous subsection, for $x = 0$ to be a stable equilibrium point we require both

$$f(0) = 0, \tag{4.24}$$

and

$$\frac{df(0)}{dx} < 0. \tag{4.25}$$

Now, our object obeys Newton's second law of motion,

$$m\,\frac{d^2x}{dt^2} = f(x). \tag{4.26}$$

Let us assume that it always stays fairly close to its equilibrium position. In this case, to a good approximation, we can represent $f(x)$ as a truncated Taylor series about this position. In other words,

$$f(x) \simeq f(0) + \frac{df(0)}{dx}\,x + O(x^2). \tag{4.27}$$

However, according to (4.24) and (4.25), the above expression can be written

$$f(x) \simeq -m\,\omega_0^2\,x, \tag{4.28}$$

where $df(0)/dx = -m\,\omega_0^2$. Hence, we conclude that our object satisfies the following approximate equation of motion,

$$\frac{d^2x}{dt^2} + \omega_0^2\,x \simeq 0, \tag{4.29}$$

provided that it does not stray too far from its equilibrium position ($x = 0$).

Equation (4.29) is called the *simple harmonic equation*, and governs the motion of *all* one-dimensional conservative systems which are slightly perturbed from some stable equilibrium point. The solution of Eq. (4.29) is well-known:

$$x(t) = a\,\sin(\omega_0\,t - \phi_0). \tag{4.30}$$

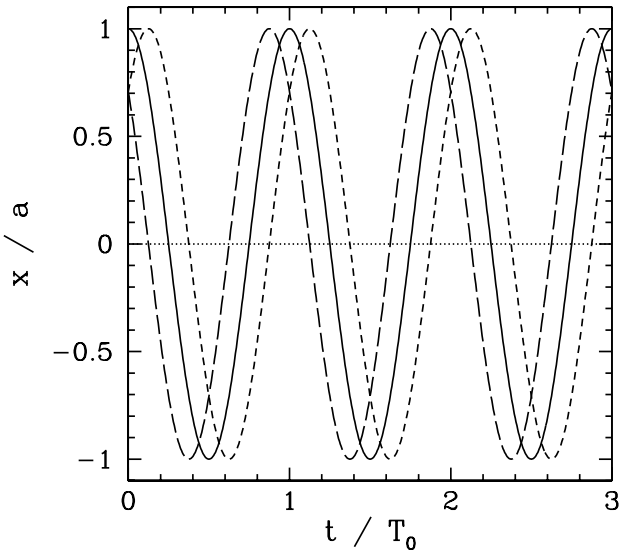

Figure 18:

The pattern of motion described by above equation, which is called *simple harmonic motion*, is *periodic* in time, with repetition period $T_0 = 2\pi/\omega_0$, and *oscillates* between $x = \pm a$. Here, a is called the *amplitude* of the motion. The parameter ϕ_0, known as the *phase angle*, simply shifts the pattern of motion backward and forward in time. Figure 18 shows examples of simple harmonic motion, Here, $\phi_0 = 0$, $+\pi/4$, and $-\pi/4$ correspond to the solid, short-dashed, and long dashed-curves, respectively.

Note that the frequency, ω_0—and, hence, the period, T_0—of simple harmonic motion is determined by the parameters in the simple harmonic equation, (4.29). However, the amplitude, a, and the phase angle, ϕ_0, are the two constants of integration of this second-order differential equation, and are, thus, determined by the initial conditions: *i.e.*, by the object's initial displacement and velocity.

Now, from Eqs. (4.1) and (4.28), the potential energy of our object at position x is approximately

$$U(x) \simeq \frac{1}{2}\, m\, \omega_0^2\, x^2. \tag{4.31}$$

Hence, the total energy is written

$$E = K + U = \frac{1}{2}\,m\left(\frac{dx}{dt}\right)^2 + \frac{1}{2}\,m\,\omega_0^2\,x^2, \tag{4.32}$$

giving

$$E = \frac{1}{2}\,m\,\omega_0^2\,a^2\,\cos^2(\omega_0\,t-\phi_0) + \frac{1}{2}\,m\,\omega_0^2\,a^2\,\sin^2(\omega_0\,t-\phi_0) = \frac{1}{2}\,m\,\omega_0^2\,a^2, \tag{4.33}$$

where use has been made of Eq. (4.30), and the trigonometric identity $\cos^2\theta + \sin^2\theta \equiv 1$. Note that the total energy is *constant* in time, as is to be expected for a conservative system, and is proportional to the *amplitude squared* of the motion.

4.5 Damped oscillatory motion

According to Eq. (4.30), a one-dimensional conservative system which is slightly perturbed from a stable equilibrium point (and then left alone) oscillates about this point with a fixed frequency and a constant amplitude. In other words, the oscillations never die away. This is not very realistic, since we know that, in practice, if we slightly perturb a dynamical system (such as a pendulum) from a stable equilibrium point then it will indeed oscillate about this point, but these oscillations will eventually die away due to frictional effects, which are present in all real dynamical systems. In order to model frictional effects, we need to include some sort of frictional drag force in our perturbed equation of motion, (4.29).

The most common model for a frictional drag force is one which is always directed in the *opposite direction* to the instantaneous velocity of the object upon which it acts, and is *directly proportional* to the magnitude of this velocity. Let us adopt this model. So, our drag force can be written

$$f_{drag} = -2\,m\,\nu\,\frac{dx}{dt}, \tag{4.34}$$

where ν is a positive constant. Including such a force in our perturbed equation of motion, (4.29), we obtain

$$\frac{d^2x}{dt^2} + 2\,\nu\,\frac{dx}{dt} + \omega_0^2\,x = 0. \tag{4.35}$$

Thus, the positive constant ν parameterizes the strength of the frictional damping in our dynamical system.

Equation (4.35) is a linear, second-order, ordinary differential equation, which we suspect possesses oscillatory solutions. There is a standard trick for solving such an equation. We search for complex oscillatory solutions of the form

$$x = a\,\mathrm{e}^{-\mathrm{i}\,\omega\,t}, \tag{4.36}$$

where both ω and a are, in general, complex. Of course, the physical solution is the *real part* of the above expression. Note that this method of solution is only appropriate for *linear* differential equations. Incidentally, the method works because

$$\mathrm{Re}[\mathcal{L}(x)] \equiv \mathcal{L}(\mathrm{Re}[x]), \tag{4.37}$$

where x is a complex variable, and $\mathcal{L}$ some real linear differential operator which acts on this variable.

Substituting Eq. (4.36) into Eq. (4.35), we obtain

$$a\left[-\omega^2 - \mathrm{i}\,2\,\nu\,\omega + \omega_0^2\right]\mathrm{e}^{-\mathrm{i}\,\omega\,t} = 0, \tag{4.38}$$

which reduces to the following quadratic equation for ω:

$$\omega^2 + \mathrm{i}\,2\,\nu\,\omega - \omega_0^2 = 0. \tag{4.39}$$

The solution to this equation is

$$\omega_\pm = -\mathrm{i}\,\nu \pm \sqrt{\omega_0^2 - \nu^2}. \tag{4.40}$$

Thus, the most general physical solution to Eq. (4.35) is

$$x(t) = \mathrm{Re}\left[a_+\,\mathrm{e}^{-\mathrm{i}\,\omega_+\,t} + a_-\,\mathrm{e}^{-\mathrm{i}\,\omega_-\,t}\right], \tag{4.41}$$

where $a_\pm$ are two arbitrary complex constants.

We can distinguish three different cases. In the first case, $\nu < \omega_0$, and the motion is said to be *underdamped*. The most general solution is written

$$x(t) = x_0\,\mathrm{e}^{-\nu\,t}\,\cos(\omega_r\,t) + \left(\frac{v_0 + \nu\,x_0}{\omega_r}\right)\mathrm{e}^{-\nu\,t}\,\sin(\omega_r\,t), \tag{4.42}$$

where $\omega_r = \sqrt{\omega_0^2 - \nu^2}$, $x_0 = x(0)$, and $v_0 = dx(0)/dt$. It can be seen that the solution oscillates at some real frequency, ω_r, which is somewhat less than the natural frequency of oscillation of the undamped system, ω_0, but also *decays* exponentially in time at a rate proportional to the damping coefficient, ν.

In the second case, $\nu = \omega_0$, and the motion is said to be *critically damped*. The most general solution is written

$$x(t) = [x_0\,(1 + \omega_0\,t) + v_0\,t]\,\mathrm{e}^{-\omega_0\,t}. \tag{4.43}$$

It can be seen that the solution now decays without oscillating.

In the third case, $\nu > \omega_0$, and the motion is said to be *overdamped*. The most general solution is written

$$x(t) = -\left(\frac{v_0 + \nu_-\,x_0}{\nu_+ - \nu_-}\right)\mathrm{e}^{-\nu_+\,t} + \left(\frac{v_0 + \nu_+\,x_0}{\nu_+ - \nu_-}\right)\mathrm{e}^{-\nu_-\,t}, \tag{4.44}$$

where $\nu_\pm = \nu \pm \sqrt{\nu^2 - \omega_0^2}$. It can be seen that the solution again decays without oscillating, except there are now *two* independent decay rates. The largest, ν_+, is always greater than the critically damped decay rate, ω_0, whereas the smaller, ν_-, is always less than this decay rate. This means that, in general, the critically damped solution is more rapidly damped than either the underdamped or overdamped solutions.

Figure 19 shows typical examples of underdamped (*i.e.*, $\nu = \omega_0/4$) , critically damped (*i.e.*, $\nu = \omega_0$), and overdamped (*i.e,*, $\nu = 4\,\omega_0$) solutions, calculated with the initial conditions $x_0 = 1$ and $v_0 = 0$. Here, $T_0 = 2\pi/\omega_0$. The three solutions correspond to the solid, short-dashed, and long-dashed curves, respectively.

4.6 Resonance

We have seen that when a one-dimensional dynamical system is slightly perturbed from a stable equilibrium point (and then left alone), it eventually returns to this point at a rate controlled by the amount of damping in the system. Let us now suppose that the same system is subject to *continuous*, constant amplitude, external

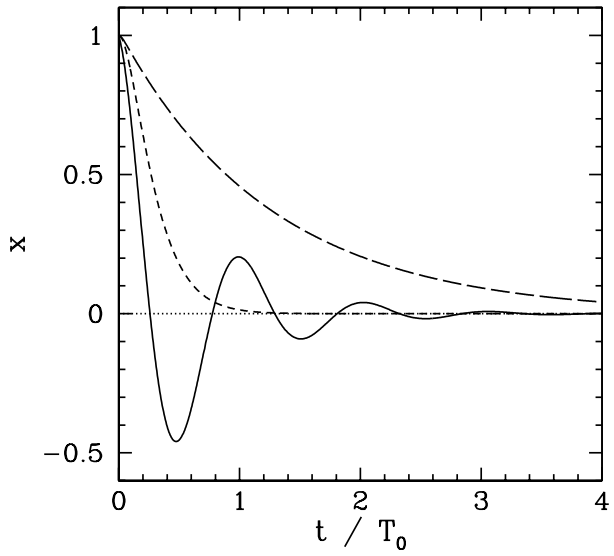

Figure 19:

forcing at some fixed frequency, ω. In this case, we would expect the system to eventually settle down to some steady oscillatory pattern of motion with the same frequency. Let us investigate the properties of this type of driven oscillation.

Suppose that our system is subject to an external force of the form

$$f_{ext}(t) = m\,\omega_0^2\,X_1\,\cos(\omega\,t). \tag{4.45}$$

Here, X_1 measures the typical ratio of the amplitude of the external force to that of the restoring force, (4.28). Incorporating the above force into our perturbed equation of motion, (4.35), we obtain

$$\frac{d^2x}{dt^2} + 2\,\nu\,\frac{dx}{dt} + \omega_0^2\,x = \omega_0^2\,X_1\,\cos(\omega\,t). \tag{4.46}$$

Let us search for a solution of the form (4.36), and represent the right-hand side of the above equation as $\omega_0^2\,X_1\,\exp(-\mathrm{i}\,\omega\,t)$. It is again understood that the physical terms are the *real parts* of these expressions. Note that ω is now a real parameter. We obtain

$$a\left[-\omega^2 - \mathrm{i}\,2\,\nu\,\omega + \omega_0^2\right]\mathrm{e}^{-\mathrm{i}\,\omega\,t} = \omega_0^2\,X_1\,\mathrm{e}^{-\mathrm{i}\,\omega\,t}. \tag{4.47}$$

Hence,

$$a = \frac{\omega_0^2\,X_1}{\omega_0^2 - \omega^2 - \mathrm{i}\,2\,\nu\,\omega}. \tag{4.48}$$

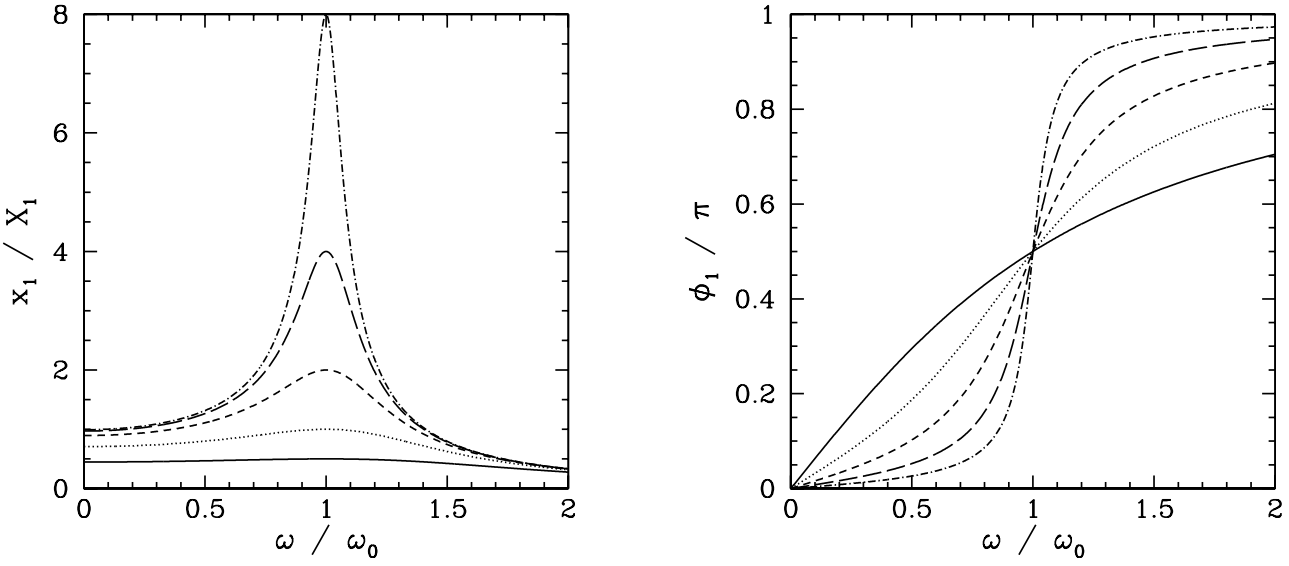

Figure 20:

In general, a is a complex quantity. Thus, we can write

$$a = x_1 \, \mathrm{e}^{\,\mathrm{i}\,\phi_1}, \tag{4.49}$$

where x_1 and ϕ_1 are both real. It follows from Eqs. (4.36), (4.48), and (4.49) that the physical solution takes the form

$$x(t) = x_1 \, \cos(\omega\, t - \phi_1), \tag{4.50}$$

where

$$x_1 = \frac{\omega_0^2 \, X_1}{[(\omega_0^2 - \omega^2)^2 + 4\,\nu^2\,\omega^2]^{1/2}}, \tag{4.51}$$

and

$$\phi_1 = \tan^{-1}\left(\frac{2\,\nu\,\omega}{\omega_0^2 - \omega^2}\right). \tag{4.52}$$

We conclude that, in response to the applied sinusoidal force, (4.45), the system executes a sinusoidal pattern of motion at the same frequency, with fixed amplitude x_1, and phase-lag ϕ_1 (with respect to the external force).

Let us investigate the variation of x_1 and ϕ_1 with the forcing frequency, ω. This is most easily done graphically. Figure 20 shows x_1 and ϕ_1 as functions of ω for various values of ν/ω_0. Here, $\nu/\omega_0 = 1$, $1/2$, $1/4$, $1/8$, and $1/16$ correspond to the solid,

dotted, short-dashed, long-dashed, and dot-dashed curves, respectively. It can be seen that as the amount of damping in the system is decreased, the amplitude of the response becomes progressively more peaked at the natural frequency of oscillation of the system, ω_0. This effect is known as *resonance*, and ω_0 is termed the *resonant frequency*. Thus, a lightly damped system (*i.e.*, $\nu \ll \omega_0$) can be driven to large amplitude by the application of a relatively small external force which oscillates at a frequency close to the resonant frequency. Note that the response of the system is in phase (*i.e.*, $\phi_1 \simeq 0$) with the external driving force for driving frequencies well below the resonant frequency, is in phase quadrature (*i.e.*, $\phi_1 = \pi/2$) at the resonant frequency, and is in anti-phase (*i.e.*, $\phi_1 \simeq \pi$) for frequencies well above the resonant frequency. It is easily demonstrated that for lightly damped systems the height of the resonance curve (*i.e.*, the x_1 versus ω curve) is inversely proportional to ν, whereas its width is directly proportional to ν, so that the area under the curve stays approximately constant as ν decreases.

4.7 Periodic driving forces

In the last section, we investigated the response of a one-dimensional dynamical system, close to a stable equilibrium point, to an external force which varies as $\cos(\omega\, t)$. Let us now examine the response of the same system to a more complicated external force.

Consider a general external force which is *periodic* in time, with period T. By analogy with Eq. (4.45), we can write such a force as

$$f_{ext}(t) = m\,\omega_0^2\, X(t), \tag{4.53}$$

where

$$X(t+T) = X(t) \tag{4.54}$$

for all t.

Now, we can represent $X(t)$ as a *Fourier series* in time. In other words, we can write

$$X(t) = \sum_{n=0}^{\infty} X_n\, \cos(n\,\omega\, t), \tag{4.55}$$

where $\omega = 2\pi/T$. By writing $X(t)$ in this form, we *automatically* satisfy the periodicity constraint (4.54). [Note that by choosing a cosine Fourier series we are limited to even functions in t: *i.e.*, $X(-t) = X(t)$. Odd functions in t can be represented by sine Fourier series, and mixed functions require a combination of cosine and sine Fourier series.] The constant coefficients X_n are known as *Fourier coefficients*. But, how do we determine these coefficients for a given functional form, $X(t)$?

Well, it follows from the periodicity of the cosine function that

$$\frac{1}{T}\int_0^T \cos(n\,\omega\,t)\,dt = \delta_{n\,0}, \tag{4.56}$$

where $\delta_{n\,n'}$ is unity if $n = n'$, and zero otherwise, and is known as the *Kronecker delta function*. Thus, integrating Eq. (4.55) over t from $t = 0$ to $t = T$, and making use of Eq. (4.56), we obtain

$$X_0 = \frac{1}{T}\int_0^T X(t)\,dt. \tag{4.57}$$

It is also easily demonstrated that

$$\frac{2}{T}\int_0^T \cos(n\,\omega\,t)\,\cos(n'\,\omega\,t)\,dt = \delta_{n\,n'}, \tag{4.58}$$

provided $n, n' > 0$. Thus, multiplying Eq. (4.55) by $\cos(n\,\omega\,t)$, integrating over t from $t = 0$ to $t = T$, and making use of Eqs. (4.56) and (4.58), we obtain

$$X_n = \frac{2}{T}\int_0^T X(t)\,\cos(n\,\omega\,t)\,dt \tag{4.59}$$

for $n > 0$. Hence, we have now determined the Fourier coefficients of the general periodic function $X(t)$.

We can incorporate the periodic external force (4.53) into our perturbed equation of motion by writing

$$\frac{d^2x}{dt^2} + 2\,\nu\,\frac{dx}{dt} + \omega_0^2\,x = \omega_0^2\sum_{n=0}^{\infty} X_n\,\mathrm{e}^{-\mathrm{i}\,n\,\omega\,t}, \tag{4.60}$$

where we are again using the convention that the physical solution corresponds to the *real part* of the complex solution. Note that the above differential equation is

linear. This means that if $x_a(t)$ and $x_b(t)$ represent two independent solutions to this equation then any linear combination of $x_a(t)$ and $x_b(t)$ is also a solution. We can exploit the linearity of the above equation to write the solution in the form

$$x(t) = \sum_{n=0}^{\infty} X_n\, a_n\, \mathrm{e}^{-\mathrm{i}\, n\, \omega\, t}, \tag{4.61}$$

where the a_n are the complex amplitudes of the solutions to

$$\frac{d^2x}{dt^2} + 2\,\nu\,\frac{dx}{dt} + \omega_0^2\, x = \omega_0^2\, \mathrm{e}^{-\mathrm{i}\, n\, \omega\, t}. \tag{4.62}$$

In other words, a_n is obtained by substituting $x = a_n\, \exp(-\mathrm{i}\, n\, \omega\, t)$ into the above equation. Hence, it follows that

$$a_n = \frac{\omega_0^2}{\omega_0^2 - n^2\,\omega^2 - \mathrm{i}\, 2\,\nu\, n\, \omega}. \tag{4.63}$$

Thus, the physical solution takes the form

$$x(t) = \sum_{n=0}^{\infty} X_n\, x_n\, \cos(n\,\omega\, t - \phi_n), \tag{4.64}$$

where

$$a_n = x_n\, \mathrm{e}^{\,\mathrm{i}\,\phi_n}, \tag{4.65}$$

and x_n and ϕ_n are real parameters. It follows from Eq. (4.63) that

$$x_n = \frac{\omega_0^2}{[(\omega_0^2 - n^2\,\omega^2)^2 + 4\,\nu^2\, n^2\, \omega^2]^{1/2}}, \tag{4.66}$$

and

$$\phi_n = \tan^{-1}\left(\frac{2\,\nu\, n\, \omega}{\omega_0^2 - n^2\,\omega^2}\right). \tag{4.67}$$

We have now fully determined the response of our dynamical system to a general periodic driving force.

As an example, suppose that the external force periodically delivers a brief kick to the system. For instance, let $X(t) = A$ for $0 \leq t \leq T/10$ and $9\,T/10 < t < T$, and $X(t) = 0$ otherwise (in the period $0 \leq t \leq T$). It follows from Eq. (4.57) and (4.59) that, in this case,

$$X_0 = 0.2\, A, \tag{4.68}$$

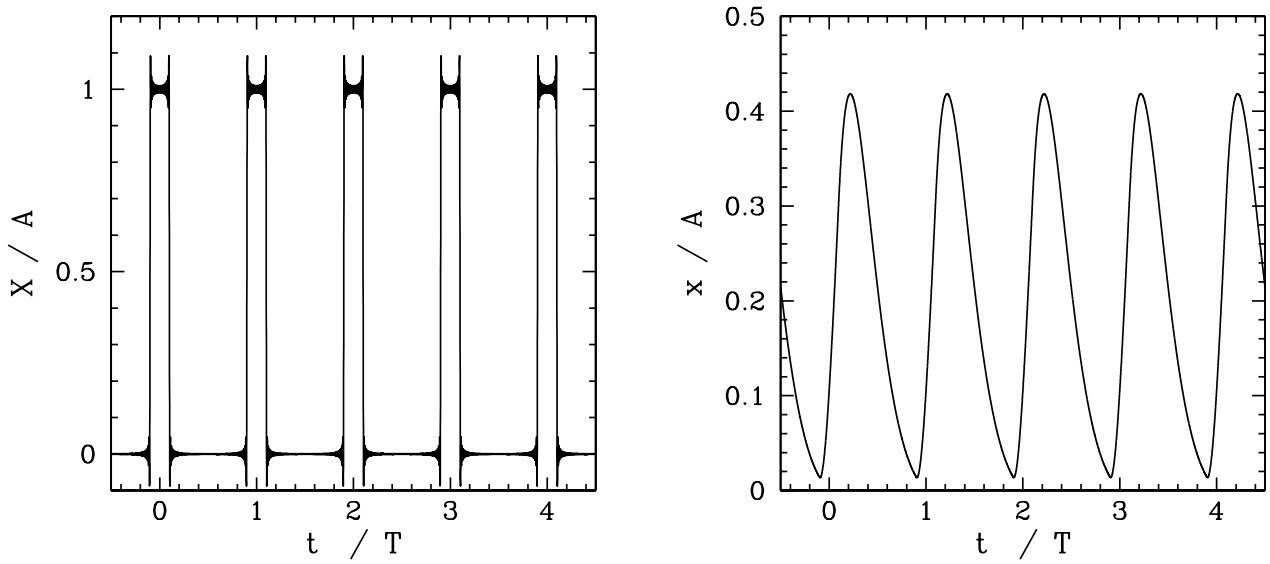

Figure 21:

and

$$X_n = \frac{2\,\sin(n\,\pi/5)\,A}{n\,\pi}, \tag{4.69}$$

for $n > 0$. Obviously, to obtain an exact solution, we would have to include every Fourier harmonic in Eq. (4.64), which is impractical. However, we can obtain a fairly accurate approximate solution by *truncating* the Fourier series (*i.e.*, by neglecting all the terms with $n > N$, where $N \gg 1$).

Figure 21 shows an example calculation in which the Fourier series is truncated after 100 terms. The parameters used in this calculation are $\omega = 1.2\,\omega_0$ and $\nu = 0.8\,\omega_0$. The left panel shows the Fourier reconstruction of the driving force, $X(t)$. The glitches at the rising and falling edges of the pulses are called *Gibbs phenomena*, and are an inevitable consequence of attempting to represent a discontinuous periodic function as a Fourier series. The right panel shows the Fourier reconstruction of the response, $x(t)$, of the dynamical system to the applied force.

4.8 Transients

We saw, in Sect. 4.6, that when a one-dimensional dynamical system, close to a stable equilibrium point, is subject to a sinusoidal external force of the form (4.45) then the equation of motion of the system is written

$$\frac{d^2x}{dt^2} + 2\,\nu\,\frac{dx}{dt} + \omega_0^2\,x = \omega_0^2\,X_1\,\cos(\omega\,t). \tag{4.70}$$

We also found that the solution to this equation which oscillates in sympathy with the applied force takes the form

$$x(t) = x_1\,\cos(\omega\,t - \phi_1), \tag{4.71}$$

where x_1 and ϕ_1 are specified in Eqs. (4.51) and (4.52), respectively. However, (4.71) is not the most general solution to Eq. (4.70). It should be clear that we can take the above solution and add to it any solution of Eq. (4.70) calculated with the right-hand side set to zero, and the result will also be a solution of Eq. (4.70). Now, we investigated the solutions to (4.70) with the right-hand set to zero in Sect. 4.5. In the underdamped regime ($\nu < \omega_0$), we found that the most general such solution takes the form

$$x(t) = A\,\mathrm{e}^{-\nu\,t}\,\cos(\omega_r\,t) + B\,\mathrm{e}^{-\nu\,t}\,\sin(\omega_r\,t), \tag{4.72}$$

where A and B are two arbitrary constants [they are in fact the constants of integration of the second-order differential equation (4.70)], and $\omega_r = \sqrt{\omega_0^2 - \nu^2}$. Thus, the most general solution to Eq. (4.70) is written

$$x(t) = A\,\mathrm{e}^{-\nu\,t}\,\cos(\omega_r\,t) + B\,\mathrm{e}^{-\nu\,t}\,\sin(\omega_r\,t) + x_1\,\cos(\omega\,t - \phi_1). \tag{4.73}$$

The first two terms on the right-hand side of the above equation are called *transients*, since they decay in time. The transients are determined by the *initial conditions*. However, if we wait long enough after setting the system into motion then the transients will always decay away, leaving the time-asymptotic solution (4.71), which is independent of the initial conditions.

As an example, suppose that we set the system into motion at time $t = 0$ with the initial conditions $x(0) = dx(0)/dt = 0$. Setting $x(0) = 0$ in Eq. (4.73), we obtain

$$A = -x_1\,\cos\phi_1. \tag{4.74}$$

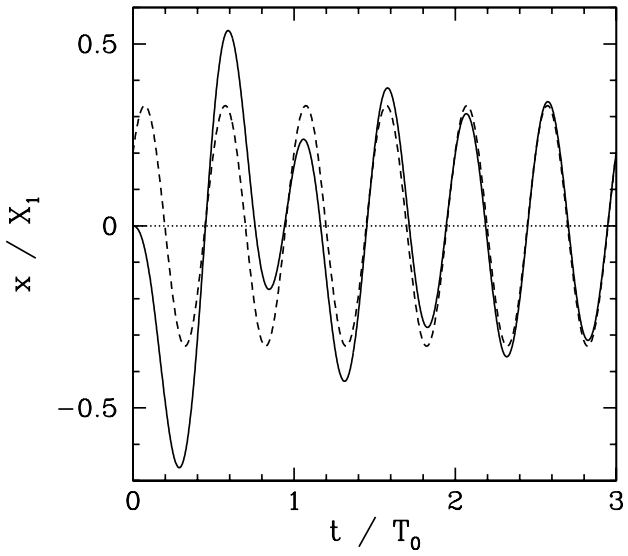

Figure 22:

Moreover, setting $dx(0)/dt = 0$ in Eq. (4.73), we get

$$B = -\frac{x_1\,(\nu\,\cos\phi_1 + \omega\,\sin\phi_1)}{\omega_r}. \tag{4.75}$$

Thus, we have now determined the constants A and B, and, hence, fully specified the solution for $t > 0$. Figure 22 shows this solution (solid curve) calculated for $\omega = 2\,\omega_0$ and $\nu = 0.2\,\omega_0$. Here, $T_0 = 2\pi/\omega_0$. The associated time-asymptotic solution (4.71) is also shown for the sake of comparison (dashed curve). It can be seen that the full solution quickly converges to the time-asymptotic solution.

4.9 The simple pendulum

Consider a mass m suspended from a light inextensible string of length l, such that the mass is free to swing from side to side in a vertical plane, as shown in Fig. 23. This setup is known as a *simple pendulum*. Let θ be the angle subtended between the string and the downward vertical. Obviously, the stable equilibrium state of the simple pendulum corresponds to the situation in which the mass is stationary, and hanging vertically down (*i.e.*, $\theta = 0$). The angular equation of motion of the

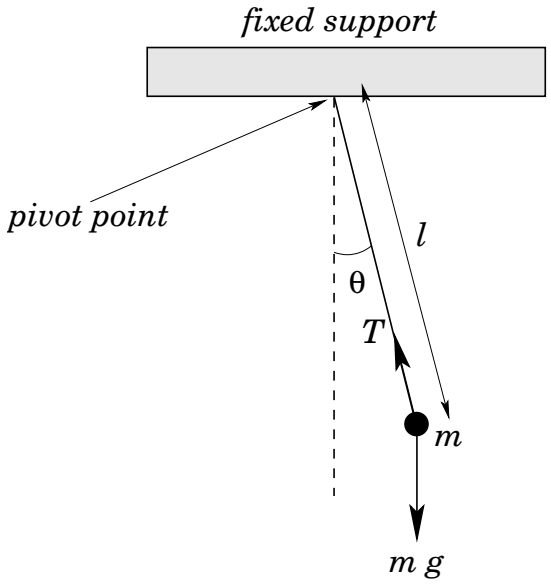

Figure 23:

pendulum is simply

$$I\,\frac{d^2\theta}{dt^2} = \tau, \tag{4.76}$$

where I is the moment of inertia of the mass, and τ is the torque acting about the pivot point. For the case in hand, given that the mass is essentially a point particle, and is situated a distance l from the axis of rotation (*i.e.*, the pivot point), it is easily seen that $I = m\,l^2$.

The two forces acting on the mass are the downward gravitational force, $m\,g$, where g is the acceleration due to gravity, and the tension, T, in the string. Note, however, that the tension makes no contribution to the torque, since its line of action clearly passes through the pivot point. From simple trigonometry, the line of action of the gravitational force passes a distance $l\,\sin\theta$ from the pivot point. Hence, the magnitude of the gravitational torque is $m\,g\,l\,\sin\theta$. Moreover, the gravitational torque is a *restoring torque*: *i.e.*, if the mass is displaced slightly from its equilibrium state (*i.e.*, $\theta = 0$) then the gravitational torque clearly acts to push the mass back toward that state. Thus, we can write

$$\tau = -m\,g\,l\,\sin\theta. \tag{4.77}$$

Combining the previous two equations, we obtain the following angular equation

of motion of the pendulum:

$$l\,\frac{d^2\theta}{dt^2} = -g\,\sin\theta. \tag{4.78}$$

Note that, unlike all of the other equations of motion which we have examined in this section, the above equation is *nonlinear*.

Let us assume, as usual, that the system does not stray very far from its equilibrium point ($\theta = 0$). If this is the case, then we can make the small angle approximation $\sin\theta \simeq \theta$, and the above equation of motion simplifies to

$$\frac{d^2\theta}{dt^2} + \omega_0^2\,\theta \simeq 0, \tag{4.79}$$

where $\omega_0 = \sqrt{g/l}$. Of course, this is just the simple harmonic equation. Hence, we can immediately write the solution as

$$\theta(t) = \theta_0\,\cos(\omega_0\,t). \tag{4.80}$$

Thus, we conclude that the pendulum swings back and forth at a fixed frequency, ω_0, which depends on l and g, but is *independent* of the amplitude, θ_0, of the motion.

Suppose, now, that we desire a more accurate solution of Eq. (4.78). One way in which we could achieve this would be to include more terms in the small angle expansion of $\sin\theta$, which is

$$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} + \cdots. \tag{4.81}$$

For instance, keeping the first two terms in this expansion, Eq. (4.78) becomes

$$\frac{d^2\theta}{dt^2} + \omega_0^2(\theta - \theta^3/6) \simeq 0. \tag{4.82}$$

By analogy with (4.80), let us try a trial solution of the form

$$\theta(t) = \vartheta_0\,\cos(\omega\,t). \tag{4.83}$$

Substituting this into Eq. (4.82), and making use of the trigonometric identity

$$\cos^3 u \equiv (3/4)\,\cos u + (1/4)\,\cos(3\,u), \tag{4.84}$$

we obtain

$$\vartheta_0 \left[\omega_0^2 - \omega^2 - (1/8)\,\omega_0^2\,\vartheta_0^2\right]\cos(\omega\,t) - (1/24)\,\omega_0^2\,\vartheta_0^3\,\cos(3\,\omega\,t) \simeq 0. \tag{4.85}$$

It is evident that the above equation cannot be satisfied for all values of t, except in the trivial case $\vartheta_0 = 0$. However, the form of this expression does suggest a better trial solution, namely

$$\theta(t) = \vartheta_0\,\cos(\omega\,t) + \alpha\,\vartheta_0^3\,\cos(3\,\omega\,t), \tag{4.86}$$

where α is $O(1)$. Substitution of this expression into Eq. (4.82) yields

$$\begin{aligned}\vartheta_0 \left[\omega_0^2 - \omega^2 - (1/8)\,\omega_0^2\,\vartheta_0^2\right]\cos(\omega\,t) +\\ \vartheta_0^3\,\left[\alpha\,\omega_0^2 - 9\,\alpha\,\omega^2 - (1/24)\,\omega_0^2\right]\,\cos(3\,\omega\,t) + O(\vartheta_0^5) \;\simeq\; 0.\end{aligned} \tag{4.87}$$

We can satisfy the above equation at all values of t, for non-zero ϑ_0, by setting the two expressions in square brackets to zero. This yields

$$\omega \simeq \omega_0\,\sqrt{1 - (1/8)\,\vartheta_0^2}, \tag{4.88}$$

and

$$\alpha \simeq -\frac{\omega_0^2}{192}. \tag{4.89}$$

Now, the amplitude of the motion is given by

$$\theta_0 = \vartheta_0 + \alpha\,\vartheta_0^3 = \vartheta_0 - \frac{\omega_0^2}{192}\,\vartheta_0^3. \tag{4.90}$$

Hence, Eq. (4.88) simplfies to

$$\omega = \omega_0\left[1 - \frac{\theta_0^2}{16} + O(\theta_0^4)\right]. \tag{4.91}$$

The above expression is only approximate, but it illustrates an important point: *i.e.*, that the frequency of oscillation of a simple pendulum is *not*, in fact, amplitude independent. Indeed, the frequency goes down slightly as the amplitude increases.

The above example illustrates how we might go about solving a nonlinear equation of motion by means of an expansion in a small parameter (in this case, the amplitude of the motion).

5 Multi-dimensional motion

5.1 Introduction

In this section, we shall use Newton's laws of motion to investigate various aspects of multi-dimensional motion.

5.2 Motion in a two-dimensional harmonic potential

Consider a particle of mass m moving in the two-dimensional harmonic potential

$$U(x,y) = \frac{1}{2}\,k\,r^2, \tag{5.1}$$

where $r = \sqrt{x^2+y^2}$, and $k > 0$. It follows that the particle is subject to a force,

$$\mathbf{f} = -\nabla U = -k\,(x,\,y) = -k\,\mathbf{r}, \tag{5.2}$$

which always points *towards* the origin, and whose magnitude increases *linearly* with increasing distance from the origin. According to Newton's second law, the equation of motion of the particle is

$$m\,\frac{d^2\mathbf{r}}{dt^2} = \mathbf{f} = -k\,\mathbf{r}. \tag{5.3}$$

When written in component form, the above equation reduces to

$$\frac{d^2x}{dt^2} = -\omega_0^2\,x, \tag{5.4}$$

$$\frac{d^2y}{dt^2} = -\omega_0^2\,y, \tag{5.5}$$

where $\omega_0 = \sqrt{k/m}$.

Since Eqs. (5.4) and (5.5) are both *simple harmonic equations*, we can immediately write their general solutions:

$$x = A\,\cos(\omega_0\,t - \phi_1), \tag{5.6}$$

$$y = B\,\cos(\omega_0\,t - \phi_2). \tag{5.7}$$

Here, A, B, ϕ_1, and ϕ_2 are arbitrary constants of integration. We can simplify the above equations slightly by shifting the origin of time (which is, after all, arbitrary): *i.e.*,

$$t \rightarrow t' + \phi_1/\omega_0. \tag{5.8}$$

Hence, we obtain

$$x = A\,\cos(\omega_0\,t'), \tag{5.9}$$

$$y = B\,\cos(\omega_0\,t' - \Delta), \tag{5.10}$$

where $\Delta = \phi_2 - \phi_1$. Note that the motion is clearly *periodic* in time, with period $T = 2\,\pi/\omega_0$. Thus, the particle must trace out some *closed trajectory* in the x-y plane. The question, now, is what does this trajectory look like as a function of the relative phase-shift, Δ, between the oscillations in the x- and y-directions?

Using standard trigonometry, we can write Eq. (5.10) in the form

$$y = B\,[\cos(\omega_0\,t')\,\cos\Delta + \sin(\omega_0\,t')\,\sin\Delta]\,. \tag{5.11}$$

Hence, using Eq. (5.9), we obtain

$$\left(\frac{y}{B} - \frac{x}{A}\,\cos\Delta\right)^2 = \sin^2(\omega_0\,t')\,\sin^2\Delta = \left(1 - \frac{x^2}{A^2}\right)\sin^2\Delta, \tag{5.12}$$

which simplifies to give

$$\frac{x^2}{A^2} - 2\,\frac{x\,y}{A\,B}\,\cos\Delta + \frac{y^2}{B^2} = \sin^2\Delta. \tag{5.13}$$

Unfortunately, the above equation is not immediately recognizable as being the equation of any particular geometric curve: *e.g.*, a circle, or an ellipse, or a parabola, *etc.*

Perhaps our problem is that we are using the wrong coordinates? Suppose that the rotate our coordinate axes about the z-axis by an angle θ, as illustrated in Fig. 3. According to Eqs. (2.90) and (2.91), our old coordinates $(x,\,y)$ are related to our new coordinates $(x',\,y')$ via

$$x = x'\,\cos\theta - y'\,\sin\theta, \tag{5.14}$$

$$y = x'\,\sin\theta + y'\,\cos\theta. \tag{5.15}$$

Let us see whether Eq. (5.13) takes a simpler form when expressed in terms of our new coordinates. Equations (5.13)–(5.15) yield

$$\begin{aligned} &x'^2\left[\frac{\cos^2\theta}{A^2}-\frac{2\,\cos\theta\,\sin\theta\,\cos\Delta}{A\,B}+\frac{\sin^2\theta}{B^2}\right]\\ &+y'^2\left[\frac{\sin^2\theta}{A^2}+\frac{2\,\cos\theta\,\sin\theta\,\cos\Delta}{A\,B}+\frac{\cos^2\theta}{B^2}\right]\\ &+x'y'\left[-\frac{2\,\sin\theta\,\cos\theta}{A^2}+\frac{2\,(\sin^2\theta-\cos^2\theta)\,\cos\Delta}{A\,B}+\frac{2\,\cos\theta\,\sin\theta}{B^2}\right] = \sin^2\Delta. \end{aligned} \tag{5.16}$$

We can simplify the above equation by setting the term involving $x'y'$ to zero. Hence,

$$-\frac{\sin 2\,\theta}{A^2}-\frac{2\,\cos 2\,\theta\,\cos\Delta}{A\,B}+\frac{\sin 2\,\theta}{B^2}=0, \tag{5.17}$$

where we have made use of some simple trigonometric identities. Thus, the $x'y'$ term disappears when θ takes the special value

$$\theta=\frac{1}{2}\,\tan^{-1}\left(\frac{2\,A\,B\,\cos\Delta}{A^2-B^2}\right). \tag{5.18}$$

In this case, Eq. (5.16) reduces to

$$\frac{x'^2}{a^2}+\frac{y'^2}{b^2}=1, \tag{5.19}$$

where

$$\frac{1}{a^2} = \frac{1}{\sin^2\Delta}\left[\frac{\cos^2\theta}{A^2}-\frac{2\,\cos\theta\,\sin\theta\,\cos\Delta}{A\,B}+\frac{\sin^2\theta}{B^2}\right], \tag{5.20}$$

$$\frac{1}{b^2} = \frac{1}{\sin^2\Delta}\left[\frac{\sin^2\theta}{A^2}+\frac{2\,\cos\theta\,\sin\theta\,\cos\Delta}{A\,B}+\frac{\cos^2\theta}{B^2}\right]. \tag{5.21}$$

Of course, we immediately recognize Eq. (5.19) as the equation of an *ellipse*, centered on the origin, whose major and minor axes are aligned along the x'- and y'-axes, and whose major and minor radii are a and b, respectively (assuming that $a>b$).

We conclude that, in general, a particle of mass m moving in the two-dimensional harmonic potential (5.1) executes a *closed elliptical orbit* (which is not necessarily

aligned along the x- and y-axes), centered on the origin, with period $T = 2\,\pi/\omega_0$, where $\omega_0 = \sqrt{k/m}$.

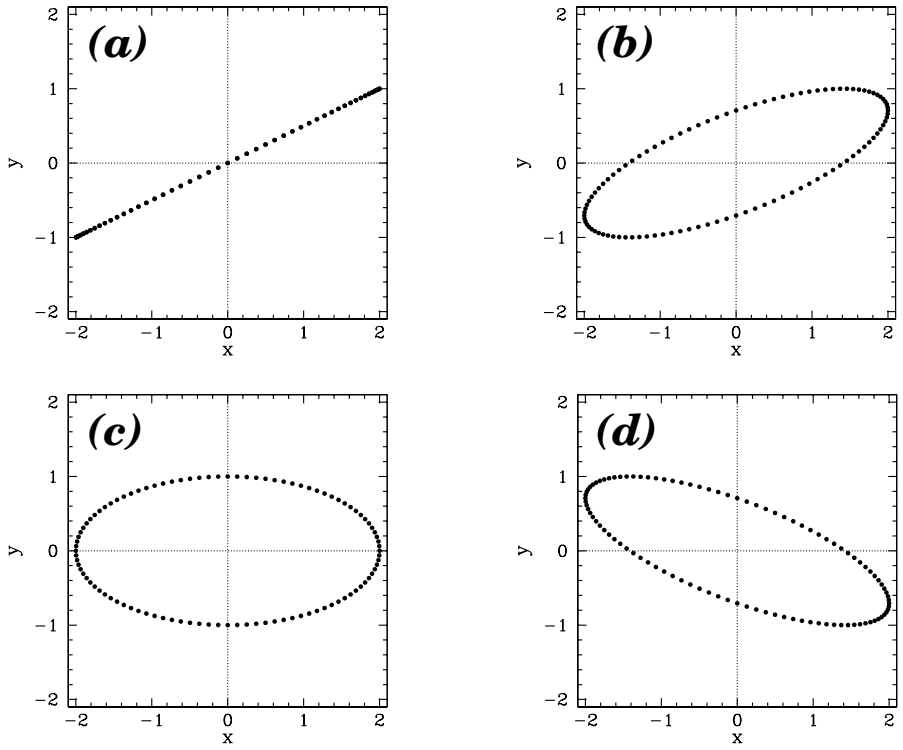

Figure 24:

Figure 24 shows some example trajectories calculated for $A = 2$, $B = 1$, and the following values of the phase difference, Δ: (a) $\Delta = 0$; (b) $\Delta = \pi/4$; (c) $\Delta = \pi/2$; (d) $\Delta = 3\pi/4$. Note that when $\Delta = 0$ the trajectory degenerates into a straight-line (which can be thought of as an ellipse whose minor radius is zero).

Perhaps, the main lesson to be learnt from the above study of two-dimensional motion in a harmonic potential is that comparatively simple patterns of motion can be made to look complicated when written in terms of ill-chosen coordinates.

5.3 Motion in crossed electric and magnetic fields

Consider a particle of mass m and electric charge q moving in the *uniform* electric and magnetic fields, $\mathbf{E}$ and $\mathbf{B}$. Suppose that the fields are "crossed" (*i.e.*,

perpendicular to one another), so that $\mathbf{E}\cdot\mathbf{B}=0$.

The force acting on the particle is given by the familiar Lorentz law:

$$\mathbf{f} = q\,(\mathbf{E}+\mathbf{v}\times\mathbf{B}), \tag{5.22}$$

where $\mathbf{v}$ is the particle's instantaneous velocity. Hence, from Newton's second law, the particle's equation of motion can be written

$$m\frac{d\mathbf{v}}{dt} = q\,(\mathbf{E}+\mathbf{v}\times\mathbf{B}). \tag{5.23}$$

It turns out that we can eliminate the electric field from the above equation by transforming to a different inertial frame. Thus, writing

$$\mathbf{v} = \frac{\mathbf{E}\times\mathbf{B}}{B^2}+\mathbf{v}', \tag{5.24}$$

Eq. (5.23) reduces to

$$m\frac{d\mathbf{v}'}{dt} = q\,\mathbf{v}'\times\mathbf{B}, \tag{5.25}$$

where we have made use of the fact that $\mathbf{E}\cdot\mathbf{B}=0$. Thus, we conclude that the addition of an electric field perpendicular to a given magnetic field simply causes all particles, irrespective of their charge or mass, to drift perpendicular to both the electric and magnetic field with the velocity

$$\mathbf{v}_{EB} = \frac{\mathbf{E}\times\mathbf{B}}{B^2}. \tag{5.26}$$

Hence, the electric field has no effect on particle motion in a frame of reference which is co-moving with the so-called *E-cross-B velocity* given above.

Let us suppose that the magnetic field is directed along the z-axis. As we have just seen, in the $\mathbf{E}\times\mathbf{B}$ frame, the particle's equation of motion reduces to Eq. (5.25), which can be written:

$$\frac{dv_x'}{dt} = \Omega\, v_y', \tag{5.27}$$

$$\frac{dv_y'}{dt} = -\Omega\, v_x', \tag{5.28}$$

$$v_z' = 0. \tag{5.29}$$

Here,

$$\Omega = \frac{q\,B}{m} \tag{5.30}$$

is the so-called *cyclotron frequency*. Equations (5.27)–(5.29) can be integrated to give

$$v_x' = v_\perp \sin(\Omega\, t), \tag{5.31}$$

$$v_y' = v_\perp \cos(\Omega\, t) \tag{5.32}$$

$$v_z' = v_\parallel, \tag{5.33}$$

where we have judiciously chosen the origin of time so as to eliminate any phase offset in the arguments of the above trigonometrical functions. According to Eqs. (5.31)–(5.33), in the $\mathbf{E}\times\mathbf{B}$ frame, charged particles gyrate at the cyclotron frequency in the plane perpendicular to the magnetic field with some fixed speed $v_\perp$, and stream parallel to the magnetic field with some fixed speed $v_\parallel$. The fact that the cyclotron frequency is positive for positively charged particles, and negative for negatively charged particles, just means that oppositely charged particles gyrate in opposite directions in the plane perpendicular to the magnetic field.

Equations (5.31)–(5.33) can be integrated to give

$$x' = -\rho \cos(\Omega\, t), \tag{5.34}$$

$$y' = \rho \sin(\Omega\, t) \tag{5.35}$$

$$z' = v_\parallel\, t, \tag{5.36}$$

where we have judiciously chosen the origin of our coordinate system so as to eliminate any constant offsets in the above equations. Here,

$$\rho = \frac{v_\perp}{\Omega} \tag{5.37}$$

is called the *Larmor radius*. Equations (5.34)–(5.36) are the equations of a *spiral* of radius ρ, aligned along the direction of the magnetic field (*i.e.*, the z-direction).

Hence, we conclude that the general motion of a charged particle in crossed electric and magnetic field is a combination of $\mathbf{E}\times\mathbf{B}$ drift [see Eq. (5.26)] and spiral motion aligned along the direction of the magnetic field. Particles drift

parallel to the magnetic field with constant speeds, and gyrate at the cyclotron frequency perpendicular to the magnetic field with constant speeds. Oppositely charged particles gyrate in opposite directions.

6 Planetary motion

6.1 Introduction

Classical mechanics was initially developed by Isaac Newton to explain the motion of the Planets around the Sun. Let us now investigate this problem.

6.2 Kepler's laws

As is well-known, Johannes Kepler was the first astronomer to correctly describe planetary motion in the Solar System (in works published between 1609 and 1619). The motion of the Planets is summed up in three simple laws:

1. The planetary orbits are all ellipses which are confocal with the Sun (*i.e.*, the Sun lies on one of the focii of the ellipses).
2. Each planet sweeps out an equal area in an equal time interval.
3. The squares of the orbital periods of the planets are proportional to the cubes of their orbital major radii.

Let us now see if we can derive Kepler's laws from Newton's laws of motion.

6.3 Newtonian gravity

As is well-know, the force which maintains the Planets in orbit around the Sun is called *gravity*, and was first correctly described by Isaac Newton (in 1687). According to Newton, any two point mass objects (or spherically symmetric objects of finite extent) exert a force of attraction on one another. This force points along the line of centers joining the objects, is directly proportional to the product of the objects' masses, and inversely proportional to the square of the distance between them. Suppose that the first object is the Sun, which is of mass M, and is located at the origin of our coordinate system. Let the second object be some planet, of

mass m, which is located at position vector $\mathbf{r}$. The gravitational force exerted on the planet by the Sun is thus written

$$\mathbf{f} = -\frac{G\,M\,m}{r^3}\,\mathbf{r}. \tag{6.1}$$

The constant of proportionality, G, is called the *gravitational constant*, and takes the value

$$G = 6.67300 \times 10^{-11}\,\mathrm{m^3\,kg^{-1}\,s^{-2}}. \tag{6.2}$$

An equal and opposite force to (6.1) acts on the Sun. However, we shall assume that the Sun is so much more massive than the planet in question that this force does not cause the Sun to shift position appreciably. Hence, the Sun will always remain at the origin of our coordinate system. Likewise, we shall neglect the gravitational forces exerted on our planet by the other planets in the Solar System compared to the much larger gravitational force exerted on it by the Sun.

Incidentally, there is something rather curious about Eq. (6.1). According to this law, the gravitational force acting on an object is directly proportional to its inertial mass. But why should inertia be related to the force of gravity? After all, inertia measures the reluctance of a given body to deviate from its preferred state of uniform motion in a straight-line, in response to some external force. What has this got to do with gravitational attraction? This question perplexed physicists for many years, and was only answered when Albert Einstein published his general theory of relativity in 1916. According to Einstein, inertial mass acts as a sort of gravitational charge since it impossible to distinguish an acceleration generated by a gravitational field from an apparent acceleration generated by being in a non-inertial frame. The assumption that these two types of acceleration are indistinguishable leads directly to all of the strange predictions of general relativity: *e.g.*, clocks in different gravitational potentials run at different rates, mass bends space, *etc.*

According to Eq. (6.1), and Newton's second law, the equation of motion of our planet takes the form

$$\frac{d^2\mathbf{r}}{dt^2} = -\frac{G\,M}{r^3}\,\mathbf{r}. \tag{6.3}$$

Note that the planetary mass, m, has cancelled out on both sides of the above equation.

6.4 Conservation laws

Now gravity is a *conservative force*. Hence, the gravitational force (6.1) can be written (see Sect. 3.4)

$$\mathbf{f} = -\nabla U, \tag{6.4}$$

where the potential energy, $U(\mathbf{r})$, of our planet in the Sun's gravitational field takes the form

$$U(\mathbf{r}) = -\frac{G\,M\,m}{r}. \tag{6.5}$$

It follows that the *total energy* of our planet is a conserved quantity (see Sect. 3.4). In other words,

$$\mathcal{E} = \frac{v^2}{2} - \frac{G\,M}{r} \tag{6.6}$$

is constant in time. Here, $\mathcal{E}$ is actually the planet's total energy *per unit mass*, and $\mathbf{v} = d\mathbf{r}/dt$.

Gravity is also a *central force*. Hence, the *angular momentum* of our planet is a conserved quantity (see Sect. 3.5). In other words,

$$\mathbf{h} = \mathbf{r} \times \mathbf{v}, \tag{6.7}$$

which is actually the planet's angular momentum *per unit mass*, is constant in time. Taking the scalar product of the above equation with $\mathbf{r}$, we obtain

$$\mathbf{h} \cdot \mathbf{r} = 0. \tag{6.8}$$

This is the equation of a *plane* which passes through the origin, and whose normal is parallel to $\mathbf{h}$. Since $\mathbf{h}$ is a constant vector, it always points in the *same* direction. We, therefore, conclude that the motion of our planet is *two-dimensional* in nature: *i.e.*, it is confined to some fixed plane which passes through the origin. Without loss of generality, we can let this plane coincide with the x-y plane.

6.5 Polar coordinates

We can determine the instantaneous position of our planet in the x-y plane in terms of standard Cartesian coordinates, (x, y), or plane polar coordinates, (r, θ),

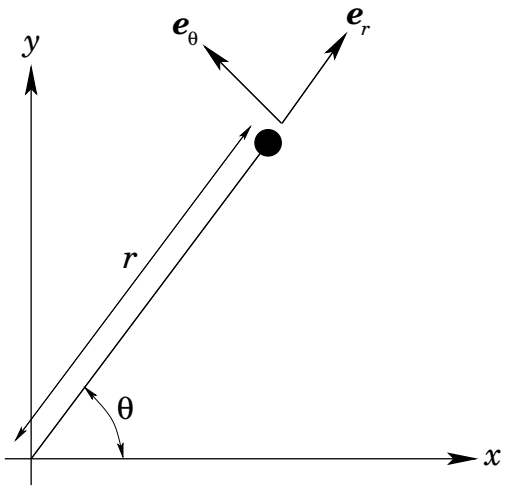

Figure 25:

as illustrated in Fig. 25. It is helpful to define two unit vectors, $\mathbf{e}_r \equiv \mathbf{r}/r$ and $\mathbf{e}_\theta \equiv \hat{\mathbf{z}} \times \mathbf{e}_r$, at the instantaneous position of the planet. The first always points radially away from the origin, whereas the second is normal to the first, in the direction of increasing θ. As is easily demonstrated, the Cartesian components of $\mathbf{e}_r$ and $\mathbf{e}_\theta$ are

$$\mathbf{e}_r = (\cos\theta, \sin\theta), \tag{6.9}$$

$$\mathbf{e}_\theta = (-\sin\theta, \cos\theta), \tag{6.10}$$

respectively.

We can write the position vector of our planet as

$$\mathbf{r} = r\,\mathbf{e}_r. \tag{6.11}$$

Thus, the planet's velocity becomes

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \dot{r}\,\mathbf{e}_r + r\,\dot{\mathbf{e}}_r, \tag{6.12}$$

where $\dot{}$ is shorthand for d/dt. Note that $\mathbf{e}_r$ has a non-zero time-derivative (unlike a Cartesian unit vector) because its direction *changes* as the planet moves around. As is easily demonstrated, from differentiating Eq. (6.9) with respect to time,

$$\dot{\mathbf{e}}_r = \dot{\theta}\,(-\sin\theta, \cos\theta) = \dot{\theta}\;\mathbf{e}_\theta. \tag{6.13}$$

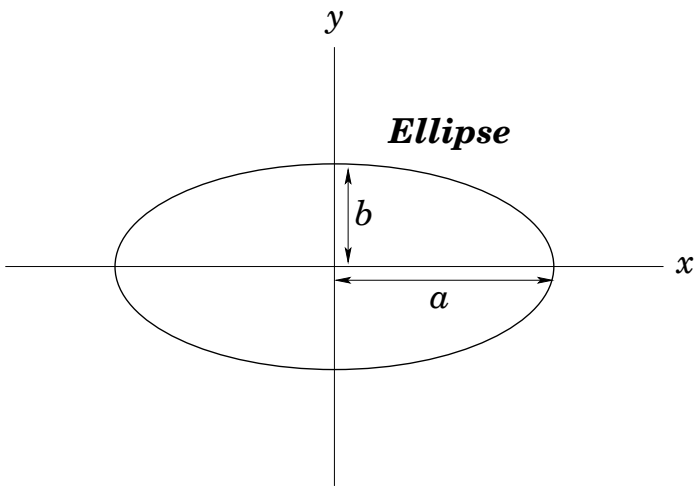

Figure 26:

Thus,

$$\mathbf{v} = \dot{r}\ \mathbf{e}_r + r\,\dot{\theta}\ \mathbf{e}_\theta. \tag{6.14}$$

Now, the planet's acceleration is written

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \frac{d^2\mathbf{r}}{dt^2} = \ddot{r}\,\mathbf{e}_r + \dot{r}\,\dot{\mathbf{e}}_r + (\dot{r}\,\dot{\theta} + r\,\ddot{\theta})\,\mathbf{e}_\theta + r\,\dot{\theta}\ \dot{\mathbf{e}}_\theta. \tag{6.15}$$

Again, $\mathbf{e}_\theta$ has a non-zero time-derivative because its direction *changes* as the planet moves around. Differentiation of Eq. (6.10) with respect to time yields

$$\dot{\mathbf{e}}_\theta = \dot{\theta}\,(-\cos\theta,\,-\sin\theta) = -\dot{\theta}\,\mathbf{e}_r. \tag{6.16}$$

Hence,

$$\mathbf{a} = (\ddot{r} - r\,\dot{\theta}^2)\,\mathbf{e}_r + (r\,\ddot{\theta} + 2\,\dot{r}\,\dot{\theta})\,\mathbf{e}_\theta. \tag{6.17}$$

It follows that the equation of motion of our planet, (6.3), can be written

$$\mathbf{a} = (\ddot{r} - r\,\dot{\theta}^2)\,\mathbf{e}_r + (r\,\ddot{\theta} + 2\,\dot{r}\,\dot{\theta})\,\mathbf{e}_\theta = -\frac{G\,M}{r^2}\,\mathbf{e}_r. \tag{6.18}$$

Since $\mathbf{e}_r$ and $\mathbf{e}_\theta$ are mutually orthogonal, we can separately equate the coefficients of both, in the above equation, to give a *radial equation of motion*,

$$\ddot{r} - r\,\dot{\theta}^2 = -\frac{G\,M}{r^2}, \tag{6.19}$$

and a *tangential equation of motion*,

$$r\,\ddot{\theta} + 2\,\dot{r}\,\dot{\theta} = 0. \tag{6.20}$$

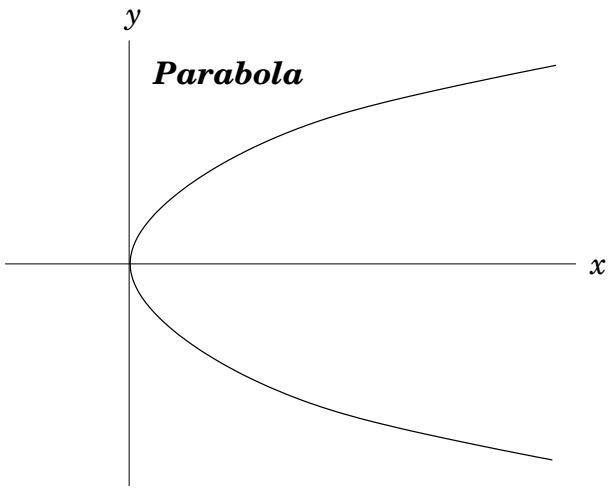

Figure 27:

6.6 Conic sections

The ellipse, the parabola, and the hyperbola are collectively known as *conic sections*, since these three types of curve can be obtained by taking various different plane sections of a right cone. It turns out that the possible solutions of Eqs. (6.19) and (6.20) are all conic sections. It is, therefore, appropriate for us to briefly review these curves.

An *ellipse*, centered on the origin, of major radius a and minor radius b, aligned along the x- and y-axes, respectively (see Fig. 26), satisfies the following well-known equation:

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1. \tag{6.21}$$

Likewise, a parabola which is aligned along the $+x$-axis, and passes through the origin (see Fig. 27), satisfies:

$$y^2-b\,x=0, \tag{6.22}$$

where $b>0$.

Finally, a hyperbola which is aligned along the $+x$-axis, and whose asymptotes intersect at the origin (see Fig. 28), satisfies:

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1. \tag{6.23}$$

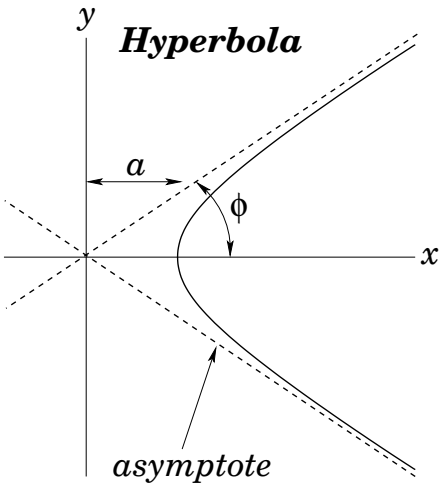

Figure 28:

Here, a is the distance of closest approach to the origin. The asymptotes subtend an angle $\phi = \tan^{-1}(b/a)$ with the x-axis.

It is not clear, at this stage, what the ellipse, the parabola, and the hyperbola have have in common. It turns out that what these three curves have in common is that they can all be represented as the locus of a movable point whose distance from a fixed point is in a constant ratio to its perpendicular distance to some fixed straight-line. Let the fixed point (which is termed the *focus* of the ellipse/parabola/hyperbola) lie at the origin, and let the fixed line correspond to $y = -d$ (with $d > 0$). Thus, the distance of a general point $(x,\ y)$ (which lies to the right of the line $y = -d$) from the origin is $r_1 = \sqrt{x^2 + y^2}$, whereas the perpendicular distance of the point from the line $y = -d$ is $r_2 = x + d$ (see Fig. 29). In plane polar coordinates, $r_1 = r$ and $r_2 = r\,\cos\theta + d$. Hence, the locus of a point for which r_1 and r_2 are in a fixed ratio satisfies the following equation:

$$\frac{r_1}{r_2} = \frac{\sqrt{x^2 + y^2}}{x + d} = \frac{r}{r\,\cos\theta + d} = e, \tag{6.24}$$

where $e \geq 0$ is a constant. When expressed in terms of plane polar coordinates, the above equation can be rearranged to give

$$r = \frac{r_c}{1 - e\,\cos\theta}, \tag{6.25}$$

where $r_c = e\,d$.

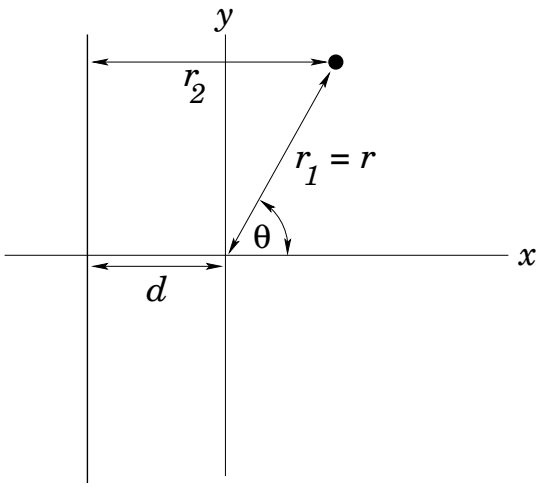

Figure 29:

When expressed in terms of Cartesian coordinates, (6.24) can be rearranged to give

$$\frac{(x-x_c)^2}{a^2}+\frac{y^2}{b^2}=1, \tag{6.26}$$

for $e<1$. Here,

$$a = \frac{r_c}{1-e^2}, \tag{6.27}$$

$$b = \frac{r_c}{\sqrt{1-e^2}} = \sqrt{1-e^2}\,a, \tag{6.28}$$

$$x_c = \frac{e\,r_c}{1-e^2} = e\,a. \tag{6.29}$$

Equation (6.26) can be recognized as the equation of an *ellipse* whose center lies at $(x_c,\ 0)$, and whose major and minor radii, a and b, are aligned along the x- and y-axes, respectively [*cf.*, Eq. (6.21)].

When again expressed in terms of Cartesian coordinates, Eq. (6.24) can be rearranged to give

$$y^2-2\,r_c\,(x-x_c)=0, \tag{6.30}$$

for $e=1$. Here, $x_c=-r_c/2$. This is the equation of a *parabola* which passes through the point $(x_c,\ 0)$, and which is aligned along the $+x$-direction [*cf.*, Eq. (6.22)].

Finally, when expressed in terms of Cartesian coordinates, Eq. (6.24) can be

rearranged to give

$$\frac{(x-x_c)^2}{a^2}-\frac{y^2}{b^2}=1, \tag{6.31}$$

for $e>1$. Here,

$$a = \frac{r_c}{e^2-1}, \tag{6.32}$$

$$b = \frac{r_c}{\sqrt{e^2-1}} = \sqrt{e^2-1}\,a, \tag{6.33}$$

$$x_c = -\frac{e\,r_c}{e^2-1} = -e\,a. \tag{6.34}$$

Equation (6.31) can be recognized as the equation of a *hyperbola* whose asymptotes intersect at $(x_c,\ 0)$, and which is aligned along the $+x$-direction. The asymptotes subtend an angle

$$\phi = \tan^{-1}\left(\frac{b}{a}\right) = \tan^{-1}(\sqrt{e^2-1}) \tag{6.35}$$

with the x-axis [*cf.*, Eq. (6.23)].

In conclusion, Eq. (6.25) is the polar equation of a *general conic section* which is *confocal with the origin*. For $e<1$, the conic section is an *ellipse*. For $e=1$, the conic section is a *parabola*. Finally, for $e>1$, the conic section is a *hyperbola*.

6.7 Kepler's second law

Multiplying our planet's tangential equation of motion, (6.20), by r, we obtain

$$r^2\,\ddot{\theta}+2\,r\,\dot{r}\,\dot{\theta}=0. \tag{6.36}$$

However, the above equation can be also written

$$\frac{d(r^2\,\dot{\theta})}{dt}=0, \tag{6.37}$$

which implies that

$$h=r^2\,\dot{\theta} \tag{6.38}$$

is constant in time. It is easily demonstrated that h is the magnitude of the vector $\mathbf{h}$ defined in Eq. (6.7). Thus, the fact that h is constant in time is equivalent to the

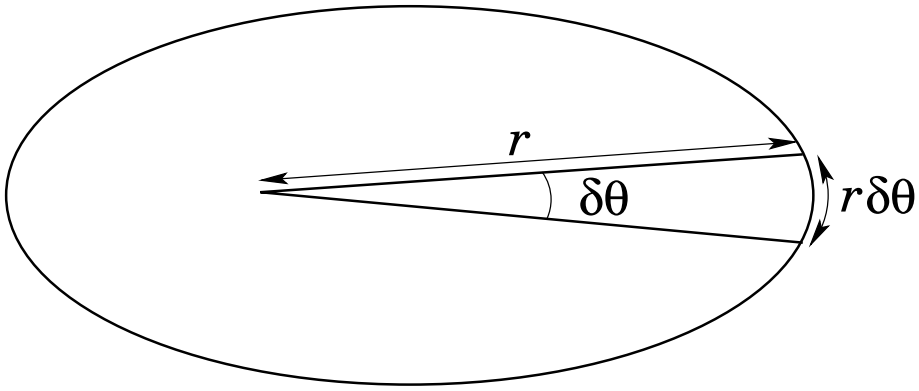

Figure 30:

statement that the angular momentum of our planet is a constant of its motion. As we have already mentioned, this is the case because gravity is a central force.

Suppose that the radius vector connecting our planet to the origin sweeps out an angle $\delta\theta$ between times t and $t+\delta t$ (see Fig. 30). The approximately triangular region swept out by the radius vector has the area

$$\delta A \simeq \frac{1}{2}\, r^2\, \delta\theta, \tag{6.39}$$

since the area of a triangle is half its base ($r\,\delta\theta$) times its height (r). Hence, the rate at which the radius vector sweeps out area is

$$\frac{dA}{dt} = \frac{1}{2}\lim_{\delta t\rightarrow 0}\frac{r^2\,\delta\theta}{\delta t} = \frac{r^2}{2}\frac{d\theta}{dt} = \frac{h}{2}. \tag{6.40}$$

Thus, the radius vector sweeps out area at a constant rate (since h is constant in time)—this is Kepler's second law. We conclude that Kepler's second law of planetary motion is a direct consequence of *angular momentum conservation*.

6.8 Kepler's first law

Our planet's radial equation of motion, (6.19), can be combined with Eq. (6.38) to give

$$\ddot{r} - \frac{h^2}{r^3} = -\frac{G\,M}{r^2}. \tag{6.41}$$

Suppose that $r = u^{-1}$. It follows that

$$\dot{r} = -\frac{\dot{u}}{u^2} = -r^2 \frac{du}{d\theta}\frac{d\theta}{dt} = -h\frac{du}{d\theta}. \tag{6.42}$$

Likewise,

$$\ddot{r} = -h\frac{d^2u}{d\theta^2}\dot{\theta} = -u^2\,h^2\frac{d^2u}{d\theta^2}. \tag{6.43}$$

Hence, Eq. (6.41) can be written

$$\frac{d^2u}{d\theta^2} + u = \frac{G\,M}{h^2}. \tag{6.44}$$

The general solution to the above equation takes the form

$$u(\theta) = \frac{G\,M}{h^2}\left[1 - e\,\cos(\theta - \theta_0)\right], \tag{6.45}$$

where e and θ_0 are arbitrary constants. Without loss of generality, we can set $\theta_0 = 0$ by rotating our coordinate system about the z-axis. Thus, we obtain

$$r(\theta) = \frac{r_c}{1 - e\,\cos\theta}, \tag{6.46}$$

where

$$r_c = \frac{h^2}{G\,M}. \tag{6.47}$$

We immediately recognize Eq. (6.46) as the equation of a conic section which is confocal with the origin (*i.e.*, with the Sun). Specifically, for $e < 1$, Eq. (6.46) is the equation of an *ellipse* which is *confocal with the Sun*. Thus, the orbit of our planet around the Sun in a confocal ellipse—this is Kepler's first law of planetary motion. Of course, a planet cannot have a parabolic or a hyperbolic orbit, since such orbits are only appropriate to objects which are ultimately able to escape from the Sun's graviational field.

6.9 Kepler's third law

We have seen that the radius vector connecting our planet to the origin sweeps out area at the constant rate $dA/dt = h/2$ [see Eq. (6.40)]. We have also seen that the

planetary orbit is an ellipse. Suppose that the major and minor radii of the ellipse are a and b, respectively. It follows that the area of the ellipse is $A = \pi\, a\, b$. Now, we expect the radius vector to sweep out the whole area of the ellipse in a single orbital period, T. Hence,

$$T = \frac{A}{(dA/dt)} = \frac{2\,\pi\, a\, b}{h}. \tag{6.48}$$

It follows from Eqs, (6.27), (6.28), and (6.47) that

$$T^2 = \frac{4\,\pi^2\, a^3}{G\, M}. \tag{6.49}$$

In other words, the *square* of the orbital *period* of our planet is proportional to the *cube* of its orbital *major radius*—this is Kepler's third law.

Note that for an elliptical orbit the closest distance to the Sun—the so-called *perihelion* distance—is [see Eqs. (6.27) and (6.46)]

$$r_p = \frac{r_c}{1+e} = a\,(1-e). \tag{6.50}$$

Likewise, the furthest distance from the Sun—the so-called *aphelion* distance—is

$$r_a = \frac{r_c}{1-e} = a\,(1+e). \tag{6.51}$$

It follows that the major radius, a, is simply the mean of the perihelion and aphelion distances,

$$a = \frac{r_p + r_a}{2}. \tag{6.52}$$

The parameter

$$e = \frac{r_a - r_p}{r_a + r_p} \tag{6.53}$$

is called the *eccentricity*, and measures the deviation of the orbit from circularity. Thus, $e = 0$ corresponds to a circular orbit, whereas $e \rightarrow 1$ corresponds to an infinitely elongated elliptical orbit.

6.10 Orbital energies

According to Eqs. (6.6) and (6.14), the total energy per unit mass of an object in orbit around the Sun is given by

$$\mathcal{E} = \frac{\dot{r}^{\,2} + r^2\,\dot{\theta}^{\,2}}{2} - \frac{G\,M}{r}. \tag{6.54}$$

It follows from Eqs. (6.38), (6.42), and (6.47) that

$$\mathcal{E} = \frac{h^2}{2}\left[\left(\frac{du}{d\theta}\right)^2 + u^2 - 2\,u\,u_c\right], \tag{6.55}$$

where $u = r^{-1}$, and $u_c = r_c^{-1}$. However, according to Eq. (6.46),

$$u(\theta) = u_c\,(1 - e\,\cos\theta). \tag{6.56}$$

The previous two equations can be combined with Eqs. (6.47) and (6.50) to give

$$\mathcal{E} = \frac{u_c^2\,h^2}{2}\,(e^2 - 1) = \frac{G\,M}{2\,r_p}\,(e - 1). \tag{6.57}$$

We conclude that *elliptical* orbits ($e < 1$) have *negative* total energies, whereas *parabolic* orbits ($e = 1$) have *zero* total energies, and *hyperbolic* orbits ($e > 1$) have *positive* total energies. This makes sense, since in a conservative system in which the potential energy at infinity is set to zero [see Eq. (6.5)] we expect *bounded* orbits to have *negative* total energies, and *unbounded* orbits to have *positive* total energies (see Sect. 4.2). Thus, elliptical orbits, which are clearly bounded, should indeed have negative total energies, whereas hyperbolic orbits, which are clearly unbounded, should indeed have positive total energies. Parabolic orbits are marginally bounded (*i.e.*, an object executing a parabolic orbit only just escapes from the Sun's gravitational field), and thus have zero total energy.

Consider an artificial satellite in an elliptical orbit around the Sun (the same considerations also apply to satellites in orbit around the Earth). At perihelion, $\dot{r} = 0$, and Eqs. (6.54) and (6.57) can be combined to give

$$\frac{v_t}{v_c} = \sqrt{1 + e}. \tag{6.58}$$

Here, $v_t = r\,\dot{\theta}$ is the satellite's tangential velocity, and $v_c = \sqrt{G\,M/r_p}$ is the tangential velocity that it would need in order to maintain a circular orbit at the perihelion distance. Likewise, at aphelion,

$$\frac{v_t}{v_c} = \sqrt{1-e}, \tag{6.59}$$

where $v_c = \sqrt{G\,M/r_a}$ is now the tangential velocity that the satellite would need in order to maintain a circular orbit at the aphelion distance.

Suppose that our satellite is initially in a circular orbit of radius r_1, and that we wish to transfer it into a circular orbit of radius r_2, where $r_2 > r_1$. We can achieve this by temporarily placing the satellite in an elliptical orbit whose perihelion distance is r_1, and whose aphelion distance is r_2. It follows, from Eq. (6.53), that the required eccentricity of the elliptical orbit is

$$e = \frac{r_2 - r_1}{r_2 + r_1}. \tag{6.60}$$

According to Eq. (6.58), we can transfer our satellite from its initial circular orbit into the temporary elliptical orbit by increasing its tangential velocity (by briefly switching on the satellite's rocket motor) by a factor

$$\alpha_1 = \sqrt{1+e}. \tag{6.61}$$

We must next allow the satellite to execute half an orbit, so that it attains its aphelion distance, and then boost the tangential velocity by a factor [see Eq. (6.59)]

$$\alpha_2 = \frac{1}{\sqrt{1-e}}. \tag{6.62}$$

The satellite will now be in a circular orbit at the aphelion distance, r_2. This process is illustrated in Fig. 31. Obviously, we can transfer our satellite from a larger to a smaller circular orbit by performing the above process in reverse. Note, finally, from Eq. (6.58), that if we increase the tangential velocity of a satellite in a circular orbit about the Sun by a factor greater than $\sqrt{2}$ then we will transfer it into a hyperbolic orbit ($e > 1$), and it will eventually escape from the Sun's gravitational field.

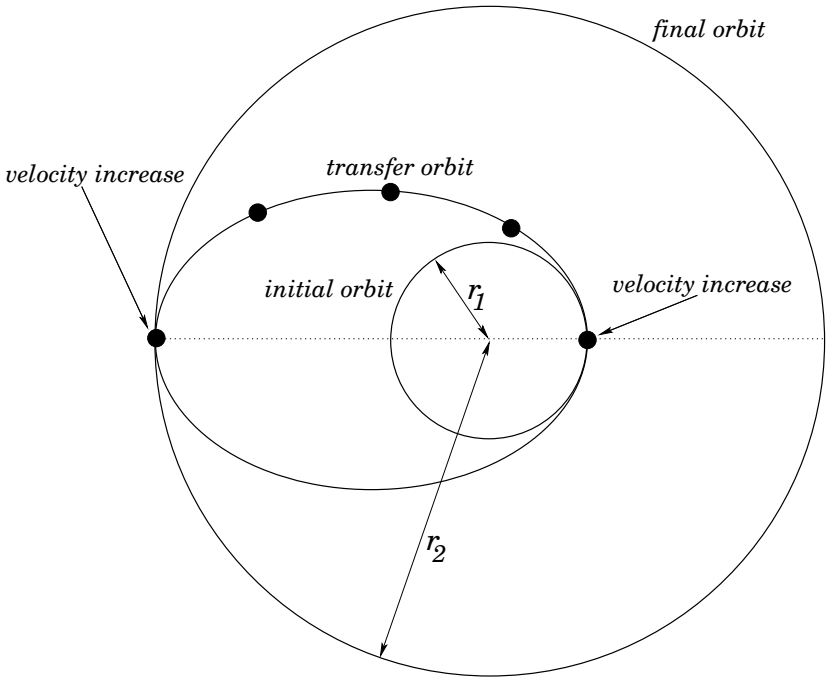

Figure 31:

6.11 Motion in a general central force-field

Consider the motion of an object in a general (attractive) central force-field characterized by the potential energy *per unit mass* function $V(r)$. Since the force-field is central, it still remains true that

$$h = r^2\,\dot{\theta} \tag{6.63}$$

is a constant of the motion. As is easily demonstrated, Eq. (6.44) generalizes to

$$\frac{d^2u}{d\theta^2} + u = -\frac{1}{h^2}\frac{dV}{du}, \tag{6.64}$$

where $u = r^{-1}$. Thus, for the case of the standard gravitational potential, $V = -G\,M\,u$, the above equation reduces to Eq. (6.44).

Suppose, for instance, that we wish to find the potential $V(r)$ which causes an object to execute the spiral orbit

$$r = r_0\,\theta^{\,2}. \tag{6.65}$$

Substitution of $u = (r_0\,\theta^2)^{-1}$ into Eq. (6.64) yields

$$\frac{dV}{du} = -h^2\left(6\,r_0\,u^2 + u\right). \tag{6.66}$$

Integrating, we obtain

$$V(u) = -h^2\left(2\,r_0\,u^3 + \frac{u^2}{2}\right), \tag{6.67}$$

or

$$V(r) = -h^2\left(\frac{2\,r_0}{r^3} + \frac{1}{2\,r^2}\right). \tag{6.68}$$

In other words, the spiral pattern (6.65) is obtained from a mixture of an inverse-square and inverse-cube potential.

6.12 Motion in a nearly circular orbit

In principle, a circular orbit is possible for any attractive central force. However, not all forces result in *stable* circular orbits. Let us now consider the stability of circular orbits in a general central force-field. Equation (6.41) generalizes to

$$\ddot{r} - \frac{h^2}{r^3} = f(r), \tag{6.69}$$

where $f(r)$ is the radial force *per unit mass*. For a circular orbit, $\ddot{r} = 0$, and the above equation reduces to

$$-\frac{h^2}{r_c^3} = f(r_c), \tag{6.70}$$

where r_c is the radius of the orbit.

Let us now consider *small* departures from circularity. Let

$$x = r - r_c. \tag{6.71}$$

Equation (6.69) can be written

$$\ddot{x} - \frac{h^2}{(r_c + x)^3} = f(r_c + x). \tag{6.72}$$

Expanding the two terms involving $r_c + x$ as power series in x/r_c, and keeping all terms up to first order, we obtain

$$\ddot{x} - \frac{h^2}{r_c^3}\left(1 - 3\,\frac{x}{r_c}\right) = f(r_c) + f'(r_c)\,x, \tag{6.73}$$

where $'$ denotes a derivative. Making use of Eq. (6.70), the above equation reduces to

$$\ddot{x} + \left[-\frac{3\,f(r_c)}{r_c} - f'(r_c)\right] x = 0. \tag{6.74}$$

If the term in square brackets is *positive* then we obtain a simple harmonic equation, which we already know has bounded solutions—*i.e.*, the orbit is *stable* to small perturbations. On the other hand, if the term is square brackets is *negative* then we obtain an equation whose solutions grow exponentially in time—*i.e.*, the orbit is *unstable* to small oscillations. Thus, the stability criterion for a circular orbit of radius r_c in a central force-field characterized by a radial force (per unit mass) function $f(r)$ is

$$f(r_c) + \frac{r_c}{3}\,f'(r_c) < 0. \tag{6.75}$$

For example, consider a power-law force function of the form

$$f(c) = -c\,r^n, \tag{6.76}$$

where $c > 0$. Substituting into the above stability criterion, we obtain

$$-\,c\,r_c^{\,n} - \frac{c\,n}{3}\,r_c^{\,n} < 0, \tag{6.77}$$

or

$$n > -3. \tag{6.78}$$

We conclude that circular orbits in central force-fields which decay faster than r^{-3} are unstable. The case $n = -3$ is special, since the first-order terms in the expansion of Eq. (6.72) cancel out exactly, and it is necessary to retain the second-order terms. Doing this, it is easily demonstrated that circular orbits are also unstable for inverse-cube ($n = -3$) forces.

An *apsis* is a point in an orbit at which the radial distance, r, assumes either a *maximum* or a *minimum* value. Thus, the perihelion and aphelion points are the

apsides of planetary orbits. The angle through which the radius vector rotates in going between two consecutive apsides is called the *apsidal angle*. Thus, the apsidal angle for elliptical orbits in an inverse-square force-field is π.

For the case of stable, nearly circular orbits, we have seen that r oscillates sinusoidally about its mean value, r_c. Indeed, it is clear from Eq. (6.74) that the period of the oscillation is

$$T = \frac{2\,\pi}{[-3\,f(r_c)/r_c - f'(r_c)]^{1/2}}. \tag{6.79}$$

The apsidal angle is the amount by which θ increases in going between a maximum and a minimum of r. The time taken to achieve this is clearly $T/2$. Now $\dot{\theta} = h/r^2$, where h is a constant of the motion, and r is almost constant. Thus, $\dot{\theta}$ is approximately constant. In fact,

$$\dot{\theta} \simeq \frac{h}{r_c^2} = \left[-\frac{f(r_c)}{r_c}\right]^{1/2}, \tag{6.80}$$

where use has been made of Eq. (6.70). Thus, the apsidal angle, ψ, is given by

$$\psi = \frac{T}{2}\,\dot{\theta} = \pi\left[3 + r_c\,\frac{f'(r_c)}{f(r_c)}\right]^{-1/2} \tag{6.81}$$

For the case of power-law central forces of the form $f(r) = -c\,r^n$, where $c > 0$, the apsidal angle becomes

$$\psi = \frac{\pi}{(3+n)^{1/2}}. \tag{6.82}$$

Now, it should be clear that if an orbit is going to close on itself then the apsidal angle needs to be a *rational* fraction of $2\,\pi$. There are, in fact, only two small integer values of the power-law index, n, for which this is the case. As we have seen, for an inverse-square force law (*i.e.*, $n = -2$), the apsidal angle is π. For a linear force law (*i.e.*, $n = 1$), the apsidal angle is $\pi/2$ (see Sect. 5.2). However, for quadratic (*i.e.*, $n = 2$) or cubic (*i.e.*, $n = 3$) force laws, the apsidal angle is an *irrational* fraction of $2\,\pi$, which means that non-circular orbits in such force-fields never close on themselves.

Let us, finally, calculate the apsial angle for a nearly circular orbit of radius r_c in a slightly modified (attractive) inverse-square force law of the form

$$f(r) = -\frac{k}{r^2} - \frac{\epsilon}{r^4}, \tag{6.83}$$

where $\epsilon/(k\,r_c^2)$ is small. Substitution into Eq. (6.81) yields

$$\psi = \pi \left[3 + r_c\,\frac{2\,k\,r_c^{-3} + 4\,\epsilon\,r_c^{-5}}{-k\,r_c^{-2} - \epsilon\,r_c^{-4}}\right]^{-1/2} = \pi\left[3 - 2\,\frac{1 + 2\,\epsilon/(k\,r_c^2)}{1 + \epsilon/(k\,r_c^2)}\right]^{-1/2}. \tag{6.84}$$

Expanding to first-order in $\epsilon/(k\,r_c^2)$, we obtain

$$\psi \simeq \pi\left(3 - 2\,[1 + \epsilon/(k\,r_c^2)]\right)^{-1/2} = \pi\left[1 - 2\,\epsilon/(k\,r_c^2)\right]^{-1/2} \simeq \pi\,[1 + \epsilon/(k\,r_c^2)]. \tag{6.85}$$

We conclude that if $\epsilon > 0$ then the perihelion (or aphelion) of the orbit *advances* by an angle $2\,\epsilon/(k\,r_c^2)$ every rotation period. Likewise, if $\epsilon < 0$ then the perihelion (or aphelion) *regresses*. For a given planet, the gravitation influence of the other planets in the Solar System can be approximated as a small $1/r^4$ correction to the Sun's gravitational field. We, therefore, conclude that the gravitational perturbation due to the other planets will cause the perihelion of the given planet to advance (or regress) by a small amount every orbital period. This effect is particularly large for Mercury, whose perihelion is observed to advance by 574 arc seconds per century. The gravitational influence of the other planets in the Solar System is responsible for 531 arc seconds of this advance. The remaining 41 arc seconds is due to general relativistic corrections to Newtonian gravity, which also give rise to a small $1/r^4$ correction to the Sun's gravitational field.

7 Two-body dynamics

7.1 Introduction

In this section, we shall investigate the dynamics of systems consisting of two freely moving, interacting, point mass objects.

7.2 Reduced mass

Suppose that our first object is of mass m_1, and is located at position vector $\mathbf{r}_1$. Likewise, our second object is of mass m_2, and is located at position vector $\mathbf{r}_2$. Let the first object exert a force $\mathbf{f}_{21}$ on the second. By Newton's third law, the second object exerts an equal and opposite force, $\mathbf{f}_{12} = -\mathbf{f}_{21}$, on the first. Suppose that there are no other forces in the problem. The equations of motion of our two objects are thus

$$m_1 \frac{d^2\mathbf{r}_1}{dt^2} = -\mathbf{f}, \tag{7.1}$$

$$m_2 \frac{d^2\mathbf{r}_2}{dt^2} = \mathbf{f}, \tag{7.2}$$

where $\mathbf{f} = \mathbf{f}_{21}$.

Now, the center of mass of our system is located at

$$\mathbf{r}_{cm} = \frac{m_1\,\mathbf{r}_1 + m_2\,\mathbf{r}_2}{m_1 + m_2}. \tag{7.3}$$

Hence, we can write

$$\mathbf{r}_1 = \mathbf{r}_{cm} - \frac{m_2}{m_1 + m_2}\,\mathbf{r}, \tag{7.4}$$

$$\mathbf{r}_2 = \mathbf{r}_{cm} + \frac{m_1}{m_1 + m_2}\,\mathbf{r}, \tag{7.5}$$

where $\mathbf{r} = \mathbf{r}_2 - \mathbf{r}_1$. Substituting the above two equations into Eqs. (7.1) and (7.2), and making use of the fact that the center of mass of an isolated system *does not*

accelerate (see Sect. 3.5), we find that both equations yield

$$\mu \frac{d^2\mathbf{r}}{dt^2} = \mathbf{f}, \tag{7.6}$$

where

$$\mu = \frac{m_1\, m_2}{m_1 + m_2} \tag{7.7}$$

is called the *reduced mass*. Hence, we have effectively converted our original two-body problem into an equivalent one-body problem. In the equivalent problem, the force $\mathbf{f}$ is the *same* as that acting on both objects in the original problem (modulo a minus sign). However, the mass, μ, is *different*, and is less than either of m_1 or m_2 (which is why it is called the "reduced" mass).

7.3 Binary star systems

Approximately half of the stars in our galaxy are members of so-called *binary star systems*. Such systems consist of two stars orbiting about their common center of mass. The distance separating the stars is always much less than the distance to the nearest neighbour star. Hence, a binary star system can be treated as a two-body dynamical system to a very good approximation.

In a binary star system, the gravitational force which the first star exerts on the second is

$$\mathbf{f} = -\frac{G\, m_1\, m_2}{r^3}\, \mathbf{r}, \tag{7.8}$$

where $\mathbf{r} = \mathbf{r}_2 - \mathbf{r}_1$. As we have seen, a two-body system can be reduced to an equivalent one-body system whose equation of motion is of the form (7.6), where $\mu = m_1\, m_2/(m_1 + m_2)$. Hence, in this particular case, we can write

$$\frac{m_1\, m_2}{m_1 + m_2}\, \frac{d^2\mathbf{r}}{dt^2} = -\frac{G\, m_1\, m_2}{r^3}\, \mathbf{r}, \tag{7.9}$$

which gives

$$\frac{d^2\mathbf{r}}{dt^2} = -\frac{G\, M}{r^3}\, \mathbf{r}, \tag{7.10}$$

where

$$M = m_1 + m_2. \tag{7.11}$$

Equation (7.10) is identical to Eq. (6.3), which we have already solved. Hence, we can immediately write down the solution:

$$\mathbf{r} = (r\,\cos\theta,\, r\,\sin\theta,\, 0), \tag{7.12}$$

where

$$r = \frac{a\,(1-e^2)}{1-e\,\cos\theta}, \tag{7.13}$$

and

$$\frac{d\theta}{dt} = \frac{h}{r^2}, \tag{7.14}$$

with

$$a = \frac{h^2}{(1-e^2)\,G\,M}. \tag{7.15}$$

Here, h is a constant, and we have aligned our Cartesian axes so that the plane of the orbit coincides with the x-y plane. According to the above solution, the second star executes a Keplerian elliptical orbit, with major radius a and eccentricity e, relative to the first star, and *vice versa*. From Eq. (6.49), the period of revolution, T, is given by

$$T = \sqrt{\frac{4\,\pi^2\,a^3}{G\,M}}. \tag{7.16}$$

In the *inertial* frame of reference whose origin always coincides with the center of mass—the so-called *center of mass frame*—the position vectors of the two stars are

$$\mathbf{r}_1 = -\frac{m_2}{m_1+m_2}\,\mathbf{r}, \tag{7.17}$$

$$\mathbf{r}_2 = \frac{m_1}{m_1+m_2}\,\mathbf{r}, \tag{7.18}$$

where $\mathbf{r}$ is specified above. Figure 32 shows an example binary star orbit, in the center of mass frame, calculated with $m_1/m_2 = 0.5$ and $e = 0.2$. Here, the triangles and squares denote the positions of the first and second star, respectively. It can be seen that both stars execute elliptical orbits about their common center of mass.

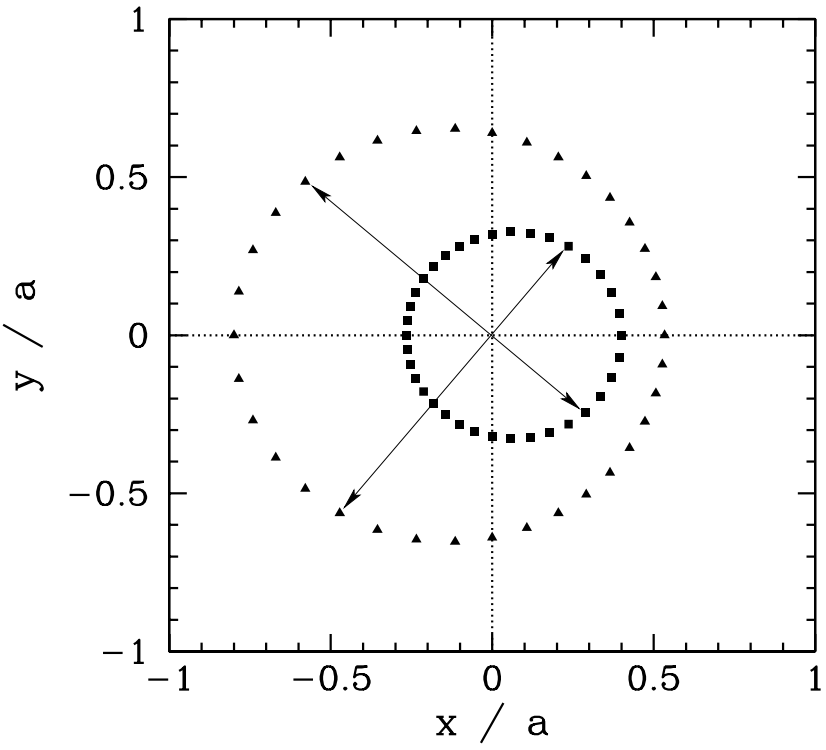

Figure 32:

Binary star systems have been very useful to astronomers, since it is possible to determine the masses of both stars in such a system by careful observation. The *sum* of the masses of the two stars, $M = m_1 + m_2$, can be determined from Eq. (7.16) after a measurement of the major radius, a (which is the mean of the greatest and smallest distance apart of the two stars during their orbit), and the orbital period, T. The *ratio* of the masses of the two stars, m_1/m_2, can be determined from Eqs. (7.17) and (7.18) by observing the fixed ratio of the relative distances of the two stars from the common center of mass about which they both appear to rotate. Obviously, given the sum of the masses, and the ratio of the masses, the individual masses themselves can then be determined.

7.4 Scattering in the center of mass frame

Let us now consider scattering due to the collision of two particles. We shall restrict our discussion to particles which interact via *conservative central forces*. It turns out that scattering looks particularly simple when viewed in the *center of mass frame*. Let us, therefore, start our investigation by considering two-particle scattering in the center of mass frame.

As before, the first particle is of mass m_1, and is located at position vector $\mathbf{r}_1$, whereas the second particle is of mass m_2, and is located at $\mathbf{r}_2$. By definition, there is *zero net linear momentum* in the center of mass frame at all times. Hence, if the first particle approaches the collision point with momentum $\mathbf{p}$, then the second must approach with momentum $-\mathbf{p}$. Likewise, after the collision, if the first particle recedes from the collision point with momentum $\mathbf{p}'$, then the second must recede with momentum $-\mathbf{p}'$ (see Fig. 33). Furthermore, since the interaction force is *conservative*, the total kinetic energy before and after the collision must be the *same*. It follows that the *magnitude* of the final momentum vector, $\mathbf{p}'$, is equal to the magnitude of the initial momentum vector, $\mathbf{p}$. Because of this, the collision event is completely specified once the angle θ through which the first particle is scattered is given. Of course, in the center of mass frame, the second particle is scattered through the same angle (see Fig. 33).

Suppose that the two particles interact via the potential $U(r)$, where r is the

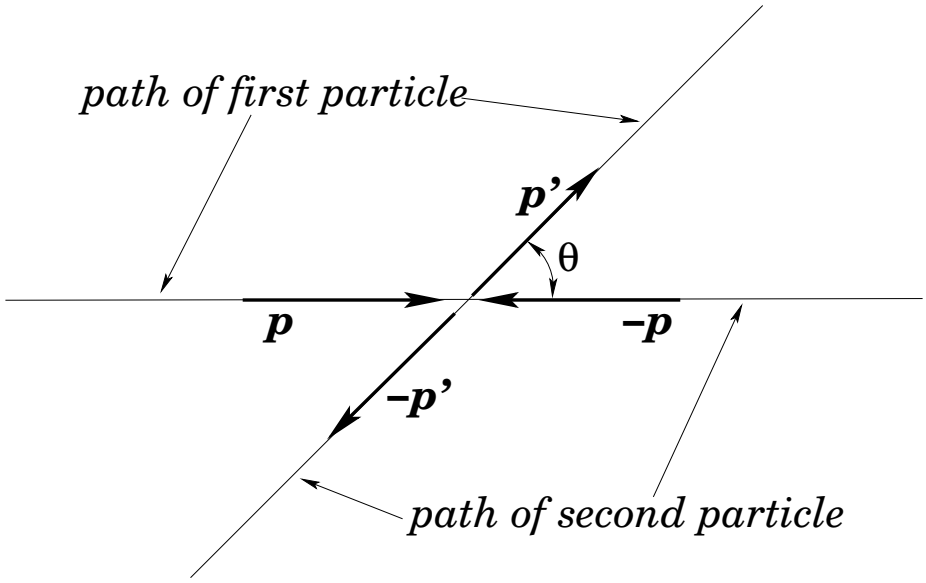

Figure 33:

distance separating the particles. As we have seen, the two-body problem sketched in Fig. 33 can be converted into the equivalent one-body problem sketched in Fig. 34. In this equivalent problem, a particle of mass $\mu = m_1\,m_2/(m_1+m_2)$ is scattered in the fixed potential $U(r)$, where r is now the distance from the origin. The vector postion $\mathbf{r}$ of the particle in the equivalent problem corresponds to the relative position vector $\mathbf{r}_2 - \mathbf{r}_1$ in the original problem. It follows that the angle θ through which the particle is scattered in the equivalent problem is the same as the scattering angle θ in the original problem.

The scattering angle, θ, is largely determined by the so-called *impact parameter*, b, which is the distance of closest approach of the two particles in the *absence* of an interaction potential. In the equivalent problem, b is the distance of closest approach to the origin in the absence of an interaction potential (see Fig. 34). If $b = 0$ then we have a head-on collision. In this case, we expect the two particles to reverse direction after colliding: *i.e.*, we expect $\theta = \pi$. Likewise, if b is large then we expect the two particles to miss one another entirely, in which case $\theta = 0$. It follows that the scattering angle, θ, is a *decreasing* function of the impact parameter, b.

Suppose that the plane polar coordinates of the particle in the equivalent problem are (r, ϑ). Let the particle approach the origin from the direction $\vartheta = 0$, and

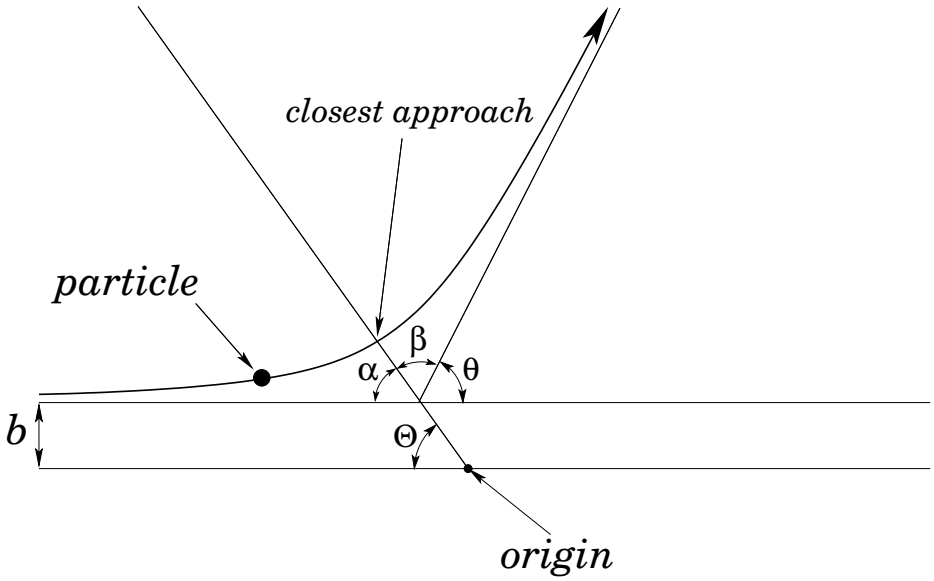

Figure 34:

attain its closest distance to the origin when $\vartheta = \Theta$. From symmetry, the angle α in Fig. 34 is equal to the angle β. However, from simple geometry $\alpha = \Theta$. Hence,

$$\theta = \pi - 2\,\Theta. \tag{7.19}$$

Now, by analogy with Eq. (6.55), the conserved total energy E in the equivalent problem, which can easily be shown to be the same as the total energy in the original problem, is given by

$$E = \frac{\mu\, h^2}{2}\left[\left(\frac{du}{d\vartheta}\right)^2 + u^2\right] + U(u), \tag{7.20}$$

where $u = r^{-1}$, and h is the angular momentum per unit mass in the equivalent problem. It is easily seen that

$$h = b\, v_\infty = b\left(\frac{2\,E}{\mu}\right)^{1/2}, \tag{7.21}$$

where v_∞ is the approach velocity in the equivalent problem at large r. It follows that

$$E = E\, b^2\left[\left(\frac{du}{d\vartheta}\right)^2 + u^2\right] + U(u). \tag{7.22}$$

The above equation can be rearranged to give

$$\frac{d\vartheta}{du} = \frac{b}{\sqrt{1 - b^2\,u^2 - U(u)/E}}. \tag{7.23}$$

Integration yields

$$\Theta = \int_0^{u_{max}} \frac{b\,du}{\sqrt{1 - b^2\,u^2 - U(u)/E}}. \tag{7.24}$$

Here, $u_{max} = 1/r_{min}$, where r_{min} is the distance of closest approach. Since, by symmetry, $(du/d\vartheta)_{u_{max}} = 0$, it follows from Eq. (7.22) that

$$1 - b^2\,u_{max}^2 - U(u_{max})/E = 0. \tag{7.25}$$

Equations (7.19) and (7.24) enable us to calculate the function $b(\theta)$ for a given interaction potential, $U(r)$, and a given energy, E, of the two particles in the center of mass frame. The function $b(\theta)$ tells us which impact parameter corresponds to which scattering angle, and *vice versa*.

Instead of two particles, suppose that we now have two counter-propagating *beams* of identical particles (with the same properties as the two particles described above) which scatter one another via binary collisions. What is the angular distribution of the scattered particles? Consider pairs of particles whose impact parameters lie in the range b to $b+db$. These particles are scattered in such a manner that their scattering angles lie in the range θ to $\theta + d\theta$, where θ is determined from inverting the function $b(\theta)$, and

$$d\theta = \frac{db}{|db(\theta)/d\theta|}. \tag{7.26}$$

Incidentally, we must take the modulus of $db(\theta)/d\theta$ because $b(\theta)$ is a decreasing function of θ. Assuming, as seems reasonable, that the scattering is azimuthally symmetric, the range of *solid angle* into which the particles are scattered is

$$d\Omega = 2\pi\,\sin\theta\,d\theta = \frac{2\pi\,\sin\theta\,db}{|db/d\theta|} \tag{7.27}$$

Finally, the *cross-sectional area* of the *annulus* through which incoming particles must pass if they are to have impact parameters in the range b to $b + db$ is

$$d\sigma = 2\pi\,b\,db. \tag{7.28}$$

The previous two equations allow us to define the *differential scattering cross-section*:

$$\frac{d\sigma}{d\Omega} = \frac{b}{\sin\theta}\left|\frac{db}{d\theta}\right| \tag{7.29}$$

The differential scattering cross-section has units of area per steradian, and specifies the effective target area for scattering into a given range of solid angle. For two *uniform* beams scattering off one another, the differential scattering cross-section thus effectively specifies the *probability* of scattering into a given range of solid angle. The *total scattering cross-section* is the integral of the differential cross-section over all solid angles,

$$\sigma = \int \frac{d\sigma}{d\Omega}\, d\Omega, \tag{7.30}$$

and measures the effective target area for scattering in *any* direction. Thus, if the flux of particles per unit area per unit time, otherwise known as the *intensity*, of the two beams is I, then the number of particles of a given type scattered per unit time is simply $I\,\sigma$.

Let us now calculate the scattering cross-section for the following very simple interaction potential:

$$U(r) = 0 \qquad \text{for } r > a, \tag{7.31}$$

$$U(r) = \infty \qquad \text{for } r \leq a. \tag{7.32}$$

This is the interaction potential of *impenetrable spheres* which only exert a force on one another when they are in physical contact (*e.g.*, billiard balls). If the particles in the first beam have radius R_1, and the particles in the second beam have radius R_2, then $a = R_1 + R_2$. In other words, the centers of two particles, one from either beam, can never be less that the distance a apart, where a is the sum of their radii (since the particles are impenetrable spheres).

Equations (7.19), (7.24), (7.31), and (7.32) yield

$$\theta = \pi - 2\int_0^{1/a} \frac{b\,du}{\sqrt{1 - b^2\,u^2}} = \pi - 2\,\sin^{-1}(b/a). \tag{7.33}$$

The above formula can be rearranged to give

$$b(\theta) = a\,\cos(\theta/2). \tag{7.34}$$

Note that

$$b\left|\frac{db}{d\theta}\right| = \frac{1}{2}\left|\frac{db^2}{d\theta}\right| = \frac{a^2}{2}\,\sin(\theta/2)\,\cos(\theta/2) = \frac{a^2}{4}\,\sin\theta. \tag{7.35}$$

Hence, Eqs. (7.29) and (7.35) yield

$$\frac{d\sigma}{d\Omega} = \frac{a^2}{4}. \tag{7.36}$$

We thus conclude that when two beams of impenetrable spheres collide, in the center of mass frame, the particles in the two beams have an equal probability of being scattered in any direction. The total scattering cross-section is

$$\sigma = \int \frac{d\sigma}{d\Omega}\,d\Omega = \pi\,a^2. \tag{7.37}$$

Obviously, this result makes a lot of sense—the total scattering cross-section for two beams of impenetrable spheres is simply the area of a circle whose radius is the sum of the radii of the two types of particles in the two beams.

Let us now consider scattering by an inverse-square interaction force whose potential takes the form

$$U(r) = \frac{k}{r}. \tag{7.38}$$

It follows from Eqs. (7.24) and (7.25) that

$$\Theta = \int_0^{u_{max}} \frac{b\,du}{\sqrt{1 - b^2\,u^2 - k\,u/E}} = \int_0^{x_{max}} \frac{dx}{\sqrt{1 - x^2 - \alpha\,x}}, \tag{7.39}$$

where $\alpha = k/(E\,b)$, and

$$1 - x_{max}^2 - \alpha\,x_{max} = 0. \tag{7.40}$$

Integration yields

$$\Theta = \frac{\pi}{2} - \sin^{-1}\left(\frac{\alpha}{\sqrt{4+\alpha^2}}\right). \tag{7.41}$$

Hence, from Eq. (7.19), we obtain

$$\theta = 2\,\sin^{-1}\left(\frac{\alpha}{\sqrt{4+\alpha^2}}\right). \tag{7.42}$$

The above equation can be rearranged to give

$$b^2 = \frac{k^2}{4\,E^2}\,\cot^2(\theta/2). \tag{7.43}$$

Hence,

$$2\,b\left|\frac{db}{d\theta}\right| = \frac{k^2}{8\,E^2}\frac{\sin\theta}{\sin^4(\theta/2)}. \tag{7.44}$$

Finally, using Eq. (7.29), we get

$$\frac{d\sigma}{d\Omega} = \frac{k^2}{16\,E^2}\frac{1}{\sin^4(\theta/2)}. \tag{7.45}$$

There are a number of things to note about the above formula. First, the scattering cross-section is proportional to k^2. This means that *repulsive* ($k > 0$) and *attractive* ($k < 0$) inverse-square-law interaction potentials of the same strength give rise to *identical* angular distributions of scattered particles. Second, the scattering cross-section is proportional to E^{-2}. This means that inverse-square-law interaction potentials are much more effective at scattering low energy, rather than high energy, particles. Finally, the differential scattering cross-section is proportional to $\sin^{-4}(\theta/2)$. This means that, with an inverse-square-law interaction potential, the overwhelming majority of "collisions" consist of *small angle* scattering events (*i.e.*, $\theta \ll 1$).

Let us now consider a specific case. Suppose that we have particles of electric charge q scattering off particles of the same charge. The interaction potential due to the Coulomb force between the particles is simply

$$U(r) = \frac{q^2}{4\pi\,\epsilon_0\,r}. \tag{7.46}$$

Thus, it follows from Eq. (7.45) [with $k = q^2/(4\pi\,\epsilon_0)$] that the differential scattering cross-section takes the form

$$\frac{d\sigma}{d\Omega} = \frac{q^4}{16\,(4\,\pi\,\epsilon_0)^2\,E^2}\frac{1}{\sin^4(\theta/2)}. \tag{7.47}$$

This very famous formula is known as the *Rutherford scattering cross-section*, since it was first derived by Earnst Rutherford for use in his celebrated α-particle scattering experiment.

Note, finally, that if we try to integrate the Rutherford formula to obtain the *total* scattering cross-section then we find that the integral is *divergent*, due to the very strong increase in $d\sigma/d\Omega$ as $\theta \rightarrow 0$. This implies that the Coulomb potential (or any other inverse-square-law potential) has an effectively *infinite* range. In practice, however, an electric charge in nature is generally surrounded by charges of the opposite sign which *shield* the Coulomb potential of the charge beyond a certain distance. This shielding effect allows the charge to have a *finite* total scattering cross-section (for the scattering of other electric charges). However, the total scattering cross-section of the charge depends (allbeit, logarithmically) on the shielding distance, and, hence, on the nature and distribution of the charges surrounding it.

7.5 Scattering in the laboratory frame

We have seen that two-particle scattering looks fairly simple when viewed in the center of mass frame. Unfortunately, we are not usually in a position to do this. In the laboratory, the most common scattering scenario is one in which the second particle is initially at rest. Let us now investigate this scenario.

Suppose that, in the center of mass frame, the first particle has velocity $\mathbf{v}_1$ before the collision, and velocity $\mathbf{v}_1'$ after the collision. Likewise, the second particle has velocity $\mathbf{v}_2$ before the collision, and $\mathbf{v}_2'$ after the collision. We know that

$$m_1\,\mathbf{v}_1 + m_2\,\mathbf{v}_2 = m_1\,\mathbf{v}_1' + m_2\,\mathbf{v}_2' = \mathbf{0} \tag{7.48}$$

in the center of mass frame. Moreover, since the collision is assumed to be elastic,

$$v_1' = v_1, \tag{7.49}$$

$$v_2' = v_2. \tag{7.50}$$

Let us transform to a new inertial frame of reference—which we shall call the *laboratory frame*—which is moving with the uniform velocity $-\mathbf{v}_2$ with respect to the center of mass frame. In the new reference frame, the first particle has initial velocity $\mathbf{V}_1 = \mathbf{v}_1 - \mathbf{v}_2$, and final velocity $\mathbf{V}_1' = \mathbf{v}_1' - \mathbf{v}_2$. Futhermore, the second particle is initially at *rest*, and has the final velocity $\mathbf{V}_2' = \mathbf{v}_2' - \mathbf{v}_2$. The

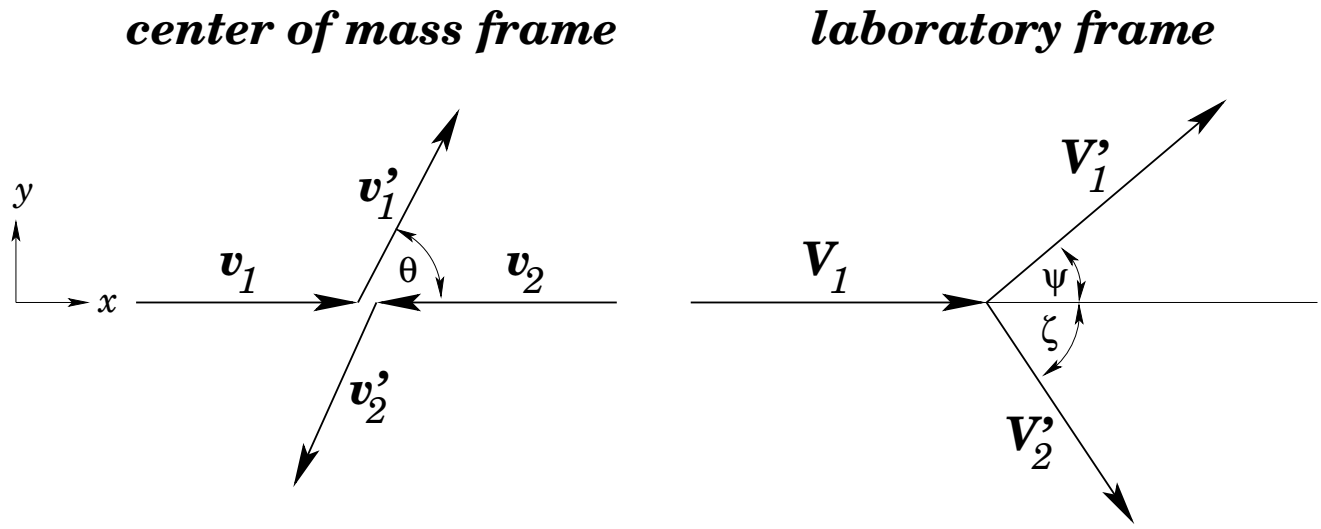

Figure 35:

relationship between scattering in the center of mass frame and scattering in the laboratory frame is illustrated in Fig. 35.

In the center of mass frame, both particles are scattered through the same angle θ. However, in the laboratory frame, the first and second particles are scattered by the (generally different) angles ψ and ζ, respectively.

Defining x- and y-axes, as indicated in Fig. 35, it is easily seen that the Cartesian components of the various velocity vectors in the two frames of reference are:

$$\mathbf{v}_1 = v_1\,(1,\,0), \tag{7.51}$$

$$\mathbf{v}_2 = (m_1/m_2)\,v_1\,(-1,\,0), \tag{7.52}$$

$$\mathbf{v}_1' = v_1\,(\cos\theta,\,\sin\theta), \tag{7.53}$$

$$\mathbf{v}_2' = (m_1/m_2)\,v_1(-\cos\theta,\,-\sin\theta), \tag{7.54}$$

$$\mathbf{V}_1 = (1+m_1/m_2)\,v_1\,(1,\,0), \tag{7.55}$$

$$\mathbf{V}_1' = v_1\,(\cos\theta + m_1/m_2,\,\sin\theta), \tag{7.56}$$

$$\mathbf{V}_2' = (m_1/m_2)\,v_1\,(1-\cos\theta,\,-\sin\theta). \tag{7.57}$$

Let E be the total energy in the center of mass frame, and $E_1 = (1/2)\,m_1\,v_1^2$ and $E_2 = (1/2)\,m_2\,v_2^2$ the kinetic energies of the first and second particles, respectively,

before the collision. Likewise, let $E_1' = (1/2)\,m_1\,v_1'^2$ and $E_2' = (1/2)\,m_2\,v_2'^2$ be the kinetic energies of the first and second particles, respectively, after the collision. Of course, $E = E_1 + E_2 = E_1' + E_2'$. Let $\mathcal{E}$ be the total energy in the laboratory frame. This is, of course, equal to the kinetic energy of the first particle before the collision. Likewise, let $\mathcal{E}_1' = (1/2)\,m_1\,V_1'^2$ and $\mathcal{E}_2' = (1/2)\,m_2\,V_2'^2$ be the kinetic energies of the first and second particles, respectively, after the collision. Of course, $\mathcal{E} = \mathcal{E}_1' + \mathcal{E}_2'$.

The following results can easily be obtained from the above definitions and Eqs. (7.51)–(7.57). First,

$$\mathcal{E} = \left(\frac{m_1 + m_2}{m_2}\right) E. \tag{7.58}$$

Hence, the total energy in the laboratory frame is always *greater* than that in the center of mass frame. In fact, it can be demonstrated that the total energy in the center of mass frame is less than the total energy in *any* other inertial frame. Second,

$$E_1 = E_1' = \left(\frac{m_2}{m_1 + m_2}\right) E, \tag{7.59}$$

$$E_2 = E_2' = \left(\frac{m_1}{m_1 + m_2}\right) E. \tag{7.60}$$

These equations specify how the total energy in the center of mass frame is distributed between the two particles. Note that this distribution is *unchanged* by the collision. Finally,

$$\mathcal{E}_1' = \left[\frac{m_1^2 + 2\,m_1\,m_2\,\cos\theta + m_2^2}{(m_1 + m_2)^2}\right] \mathcal{E}, \tag{7.61}$$

$$\mathcal{E}_2' = \left[\frac{2\,m_1\,m_2\,(1 - \cos\theta)}{(m_1 + m_2)^2}\right] \mathcal{E}. \tag{7.62}$$

These equations specify how the total energy in the laboratory frame is distributed between the two particles after the collision. Note that the energy distribution in the laboratory frame is *different* before and after the collision.

Equations (7.51)–(7.57), and some simple trigonometry, yield

$$\tan\psi = \frac{\sin\theta}{\cos\theta + m_1/m_2}, \tag{7.63}$$

and

$$\tan\zeta = \frac{\sin\theta}{1-\cos\theta} = \tan\left(\frac{\pi}{2}-\frac{\theta}{2}\right). \tag{7.64}$$

The last equation implies that

$$\zeta = \frac{\pi}{2}-\frac{\theta}{2}. \tag{7.65}$$

Differentiating Eq. (7.63) with respect to θ, we obtain

$$\frac{d\tan\psi}{d\theta} = \frac{1+(m_1/m_2)\,\cos\theta}{(\cos\theta+m_1/m_2)^2}. \tag{7.66}$$

Thus, $\tan\psi$ attains an extreme value, which can be shown to correspond to a *maximum* possible value of ψ, when the numerator of the above expression is zero: *i.e.*, when

$$\cos\theta = -\frac{m_2}{m_1}. \tag{7.67}$$

Note that it is only possible to solve the above equation when $m_1 > m_2$. If this is the case, then Eq. (7.63) yields

$$\tan\psi_{max} = \frac{m_2/m_1}{\sqrt{1-(m_2/m_1)^2}}, \tag{7.68}$$

which reduces to

$$\psi_{max} = \sin^{-1}\left(\frac{m_2}{m_1}\right). \tag{7.69}$$

Hence, we conclude that when $m_1 > m_2$ there is a *maximum* possible value of the scattering angle, ψ, in the laboratory frame. This maximum value is always less than $\pi/2$, which implies that there is *no backward scattering* (*i.e.*, $\psi > \pi/2$) at all when $m_1 > m_2$. For the special case when $m_1 = m_2$, the maximum scattering angle is $\pi/2$. However, for $m_1 < m_2$ there is no maximum value, and the scattering angle in the laboratory frame can thus range all the way to π.

Equations (7.58)–(7.65) enable us to relate the particle energies and scattering angles in the laboratory frame to those in the center of mass frame. In general, this relationship is fairly complicated. However, there are two special cases in which the relationship becomes much simpler.

The first special case is when $m_2 \gg m_1$. In this limit, it is easily seen from Eqs. (7.58)–(7.65) that the second mass is *stationary* both before and after the

collision, and that the center of mass frame *coincides* with the laboratory frame (since the energies and scattering angles in the two frames are the same). Hence, the simple analysis outlined in Sect. 7.4 is applicable in this case.

The second special case is when $m_1 = m_2$. In this case, Eq. (7.63) yields

$$\tan\psi = \frac{\sin\theta}{\cos\theta + 1} = \tan(\theta/2). \tag{7.70}$$

Hence,

$$\psi = \frac{\theta}{2}. \tag{7.71}$$

In other words, the scattering angle of the first particle in the laboratory frame is *half* of the scattering angle in the center of mass frame. The above equation can be combined with Eq. (7.65) to give

$$\psi + \zeta = \frac{\pi}{2}. \tag{7.72}$$

Thus, in the laboratory frame, the two particles move off at *right-angles* to one another after the collision. Equation (7.58) yields

$$\mathcal{E} = 2\,E. \tag{7.73}$$

In other words, the total energy in the laboratory frame is *twice* that in the center of mass frame. According to Eqs. (7.59) and (7.60),

$$E_1 = E_1' = E_2 = E_2' = \frac{E}{2}. \tag{7.74}$$

Hence, the total energy in the center of mass frame is divided *equally* between the two particles. Finally, Eqs. (7.61) and (7.62) give

$$\mathcal{E}_1' = \left(\frac{1+\cos\theta}{2}\right)\mathcal{E} = \cos^2(\theta/2)\;\mathcal{E} = \cos^2\psi\;\mathcal{E}, \tag{7.75}$$

$$\mathcal{E}_2' = \left(\frac{1-\cos\theta}{2}\right)\mathcal{E} = \sin^2(\theta/2)\;\mathcal{E} = \sin^2\psi\;\mathcal{E}. \tag{7.76}$$

Thus, in the laboratory frame, the unequal energy distribution between the two particles after the collision is simply related to the scattering angle ψ.

What is the angular distribution of scattered particles when a beam of particles of the first type scatter off stationary particles of the second type? Well, we can define a differential scattering cross-section, $d\sigma(\psi)/d\Omega'$, in the laboratory frame, where $\Omega' = 2\pi \sin\psi\, d\psi$ is an element of solid angle in this frame. Thus, $(d\sigma(\psi)/d\Omega')\, d\Omega'$ is the effective cross-sectional area in the laboratory frame for scattering into the range of scattering angles ψ to $\psi + d\psi$. Likewise, $(d\sigma(\theta)/d\Omega)\, d\Omega$ is the effective cross-sectional area in the center of mass frame for scattering into the range of scattering angles θ to $\theta + d\theta$. Note that $d\Omega = 2\pi \sin\theta\, d\theta$. However, a cross-sectional area is not changed when we transform between different inertial frames. Hence, we can write

$$\frac{d\sigma(\psi)}{d\Omega'}\, d\Omega' = \frac{d\sigma(\theta)}{d\Omega}\, d\Omega, \tag{7.77}$$

provided that ψ and θ are related via Eq. (7.63). This equation can be rearranged to give

$$\frac{d\sigma(\psi)}{d\Omega'} = \frac{d\Omega}{d\Omega'}\, \frac{d\sigma(\theta)}{d\Omega}, \tag{7.78}$$

or

$$\frac{d\sigma(\psi)}{d\Omega'} = \frac{\sin\theta}{\sin\psi}\, \frac{d\theta}{d\psi}\, \frac{d\sigma(\theta)}{d\Omega}, \tag{7.79}$$

The above equation allows us to relate the differential scattering cross-section in the laboratory frame to that in the center of mass frame. In general, this relationship is extremely complicated. However, for the special case where the masses of the two types of particles are *equal*, we have seen that $\psi = \theta/2$ [see Eq. (7.71)]. Hence, it follows from Eq. (7.79) that

$$\frac{d\sigma(\psi)}{d\Omega'} = 4\, \cos\psi\, \frac{d\sigma(\theta = 2\,\psi)}{d\Omega}. \tag{7.80}$$

Let us now consider some specific examples. We saw earlier that, in the center of mass frame, the differential scattering cross-section for *impenetrable spheres* is [see Eq. (7.36)]

$$\frac{d\sigma(\theta)}{d\Omega} = \frac{a^2}{4}, \tag{7.81}$$

where a is the sum of the radii. According to Eq. (7.80), the differential scattering cross-section (for equal mass spheres) in the laboratory frame is

$$\frac{d\sigma(\psi)}{d\Omega'} = a^2\,\cos\psi. \tag{7.82}$$

Note that this cross-section is *negative* for $\psi > \pi/2$. This just tells us that there is *no scattering* with scattering angles greater than $\pi/2$ (*i.e.*, there is no backward scattering). Comparing Eqs. (7.81) and (7.82), we can see that the scattering is *isotropic* in the center of mass frame, but appears concentrated in the forward direction in the laboratory frame. We can integrate Eq. (7.82) over all solid angles to obtain the total scattering cross-section in the laboratory frame. Note that we only integrate over angular regions where the differential scattering cross-section is *positive*. Doing this, we get

$$\sigma = \pi\,a^2, \tag{7.83}$$

which is the same as the total scattering cross-section in the center of mass frame [see Eq. (7.37)]. This is a general result. The total scattering cross-section is *frame independent*, since a cross-sectional area is not modified by switching between different frames of reference.

As we have seen, the Rutherford scattering cross-section takes the form [see Eq. (7.47)]

$$\frac{d\sigma}{d\Omega} = \frac{q^4}{16\,(4\,\pi\,\epsilon_0)^2\,E^2}\frac{1}{\sin^4(\theta/2)} \tag{7.84}$$

in the center of mass frame. It follows, from Eq. (7.80), that the Rutherford scattering cross-section (for equal mass particles) in the laboratory frame is written

$$\frac{d\sigma}{d\Omega'} = \frac{q^4}{(4\,\pi\,\epsilon_0)^2\,\mathcal{E}^2}\frac{\cos\psi}{\sin^4\psi}. \tag{7.85}$$

Here, we have made use of the fact that $\mathcal{E} = 2\,E$ for equal mass particles [see Eq. (7.73)]. Note, again, that this cross-section is negative for $\psi > \pi/2$, indicating the absence of backward scattering.

8 Non-inertial reference frames

8.1 Introduction

As we have seen (in Sect. 3), Newton's second law of motion is only valid in *inertial* frames of reference. Unfortunately, we are sometimes forced to observe motion in *non-inertial* reference frames. For instance, it is most convenient for us to observe the motions of the objects in our immediate vicinity in a reference frame which is *fixed* relative to the *surface of the Earth*. Such a frame of reference is *non-inertial* in nature, since it *accelerates* with respect to a standard inertial frame due to the Earth's daily rotation about its axis. (Note that the accelerations of the frame of reference due to the Earth's orbital motion about the Sun, or the Sun's orbital motion about the Galactic Center, *etc.*, are negligible compared to the acceleration due to the Earth's rotation.) Let us now investigate motion in a *rotating* reference frame.

8.2 Rotating reference frames

Suppose that a given object has position vector $\mathbf{r}$ in some *non-rotating inertial* reference frame. Let us observe the motion of this object in a *non-inertial* reference frame which *rotates* with *constant* angular velocity $\boldsymbol{\Omega}$ about an axis passing through the origin of the inertial frame. Suppose, first of all, that our object appears *stationary* in the rotating reference frame. Hence, in the non-rotating frame, the object's position vector $\mathbf{r}$ will appear to *precess* about the origin with angular velocity $\boldsymbol{\Omega}$. It follows, from Eq. (2.41), that in the non-rotating reference frame

$$\frac{d\mathbf{r}}{dt} = \boldsymbol{\Omega} \times \mathbf{r}. \tag{8.1}$$

Suppose, now, that our object appears to move in the rotating reference frame with instantaneous velocity $\mathbf{v}'$. It is fairly obvious that the appropriate generalization of the above equation is simply

$$\frac{d\mathbf{r}}{dt} = \mathbf{v}' + \boldsymbol{\Omega} \times \mathbf{r}. \tag{8.2}$$

Let and d/dt and d/dt' denote apparent time derivatives in the non-rotating and rotating frames of reference, respectively. Since an object which is stationary in the rotating reference frame appears to move in the non-rotating frame, it is clear that $d/dt \neq d/dt'$. Writing the apparent velocity, $\mathbf{v}'$, of our object in the rotating reference frame as $d\mathbf{r}/dt'$, the above equation takes the form

$$\frac{d\mathbf{r}}{dt} = \frac{d\mathbf{r}}{dt'} + \boldsymbol{\Omega} \times \mathbf{r}, \tag{8.3}$$

or

$$\frac{d}{dt} = \frac{d}{dt'} + \boldsymbol{\Omega}\times, \tag{8.4}$$

since $\mathbf{r}$ is a general position vector. Equation (8.4) expresses the relationship between apparent time derivatives in the non-rotating and rotating reference frames.

Operating on the general position vector $\mathbf{r}$ with the time derivative (8.4), we get

$$\mathbf{v} = \mathbf{v}' + \boldsymbol{\Omega} \times \mathbf{r}. \tag{8.5}$$

This equation relates the apparent velocity, $\mathbf{v} = d\mathbf{r}/dt$, of an object with position vector $\mathbf{r}$ in the non-rotating reference frame to its apparent velocity, $\mathbf{v}' = d\mathbf{r}/dt'$, in the rotating reference frame.

Operating twice on the position vector $\mathbf{r}$ with the time derivative (8.4), we obtain

$$\mathbf{a} = \left(\frac{d}{dt'} + \boldsymbol{\Omega}\times\right)(\mathbf{v}' + \boldsymbol{\Omega} \times \mathbf{r}), \tag{8.6}$$

or

$$\mathbf{a} = \mathbf{a}' + \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{r}) + 2\,\boldsymbol{\Omega} \times \mathbf{v}'. \tag{8.7}$$

This equation relates the apparent acceleration, $\mathbf{a} = d^2\mathbf{r}/dt^2$, of an object with position vector $\mathbf{r}$ in the non-rotating reference frame to its apparent acceleration, $\mathbf{a}' = d^2\mathbf{r}/dt'^2$, in the rotating reference frame.

Applying Newton's second law of motion in the inertial (*i.e.*, non-rotating) reference frame, we obtain

$$m\,\mathbf{a} = \mathbf{f}. \tag{8.8}$$

Here, m is the mass of our object, and $\mathbf{f}$ is the (non-fictitious) force acting on it. Note that these quantities are the *same* in both reference frames. Making use of

Eq. (8.7), the apparent equation of motion of our object in the rotating reference frame takes the form

$$m\,\mathbf{a}' = \mathbf{f} - m\,\mathbf{\Omega}\times(\mathbf{\Omega}\times\mathbf{r}) - 2\,m\,\mathbf{\Omega}\times\mathbf{v}'. \tag{8.9}$$

The last two terms in the above equation are so-called "fictitious forces". Such forces are always needed to account for motion observed in non-inertial reference frames. Let us now investigate the two fictitious forces appearing in Eq. (8.9).

8.3 Centrifugal acceleration

Let our non-rotating inertial frame be one whose origin lies at the center of the Earth, and let our rotating frame be one whose origin is fixed with respect to some point, of latitude λ, on the Earth's surface (see Fig. 36). The latter reference frame thus rotates with respect to the former (about an axis passing through the Earth's center) with an angular velocity vector, $\mathbf{\Omega}$, which points from the center of the Earth towards its North Pole, and is of magnitude

$$\Omega = \frac{2\pi}{24\ \text{hrs}} = 7.27\times10^{-5}\,\text{rad./s}. \tag{8.10}$$

Consider an object which appears stationary in our rotating reference frame: *i.e.*, an object which is stationary with respect to the Earth's surface. According to Eq. (8.9), the object's apparent equation of motion in the rotating frame takes the form

$$m\,\mathbf{a}' = \mathbf{f} - m\,\mathbf{\Omega}\times(\mathbf{\Omega}\times\mathbf{r}). \tag{8.11}$$

Let the non-fictitious force acting on our object be the force of gravity, $\mathbf{f} = m\,\mathbf{g}$. Here, the local gravitational acceleration, $\mathbf{g}$, points directly towards the center of the Earth. It follows, from the above, that the apparent gravitational acceleration in the rotating frame is written

$$\mathbf{g}' = \mathbf{g} - \mathbf{\Omega}\times(\mathbf{\Omega}\times\mathbf{R}), \tag{8.12}$$

where $\mathbf{R}$ is the displacement vector of the origin of the rotating frame (which lies on the Earth's surface) with respect to the center of the Earth. Here, we are assuming that our object is situated relatively close to the Earth's surface (*i.e.*, $\mathbf{r}\simeq\mathbf{R}$).

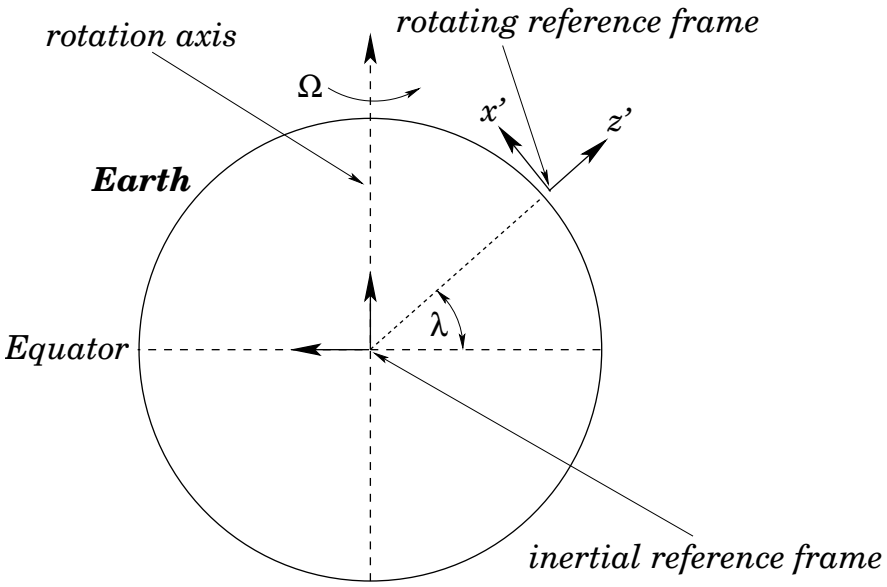

Figure 36:

It can be seen, from Eq. (8.12), that the apparent gravitational acceleration of a stationary object close to the Earth's surface has two components. First, the true gravitational acceleration, $\mathbf{g}$, of magnitude $g \sim 9.8\,\mathrm{m/s^2}$, which always points directly towards the center of the Earth. Second, the so-called *centrifugal acceleration*, $-\boldsymbol{\Omega}\times(\boldsymbol{\Omega}\times\mathbf{R})$. This acceleration is normal to the Earth's axis of rotation, and always points directly away from this axis. The magnitude of the centrifugal acceleration is $\Omega^2\,\rho = \Omega^2\,R\cos\lambda$, where ρ is the perpendicular distance to the Earth's rotation axis, and $R = 6.37\times10^6\,\mathrm{m}$ is the Earth's radius (see Fig. 37).

It is convenient to define Cartesian axes in the rotating reference frame such that the z'-axis points vertically upward, and x'- and y'-axes are horizontal, with the x'-axis pointing directly northwards, and the y'-axis pointing directly westward (see Fig. 36). The Cartesian components of the Earth's angular velocity are thus

$$\boldsymbol{\Omega} = \Omega\,(\cos\lambda,\,0,\,\sin\lambda), \tag{8.13}$$

whilst the vectors $\mathbf{R}$ and $\mathbf{g}$ are written

$$\mathbf{R} = (0,\,0,\,R), \tag{8.14}$$

$$\mathbf{g} = (0,\,0,\,-g), \tag{8.15}$$

respectively. It follows that the Cartesian coordinates of the apparent gravitational

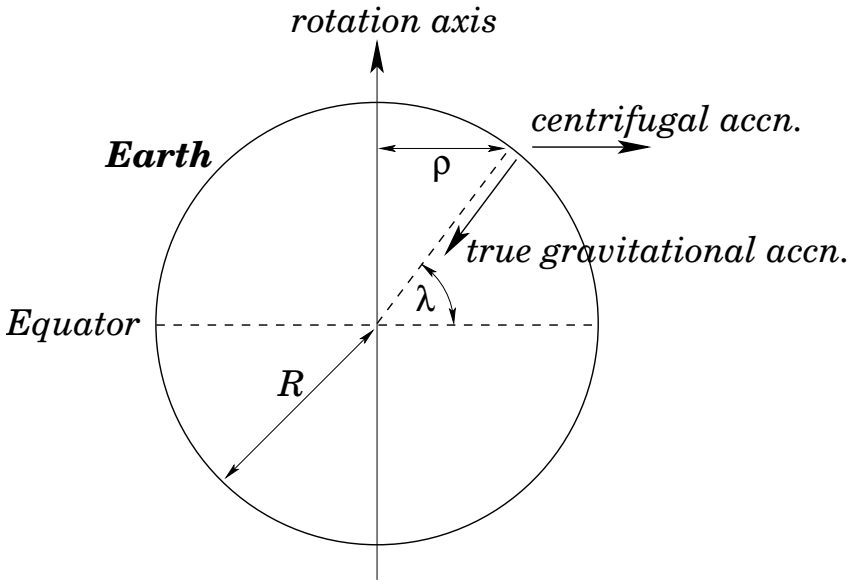

Figure 37:

acceleration, (8.12), are

$$\mathbf{g}' = \left(-\Omega^2\, R\, \cos\lambda\, \sin\lambda,\, 0,\, -g + \Omega^2\, R\, \cos^2\lambda\right). \tag{8.16}$$

The magnitude of this acceleration is approximately

$$g' \simeq g - \Omega^2\, R\, \cos^2\lambda \simeq 9.8 - 0.034\, \cos^2\lambda\ \ \mathrm{m/s^2}. \tag{8.17}$$

According to the above equation, the centrifugal acceleration causes the magnitude of the apparent gravitational acceleration on the Earth's surface to *vary* by about 0.3%, being largest at the Poles, and smallest at the Equator. This variation in apparent gravitational acceleration, due (ultimately) to the Earth's rotation, causes the Earth itself to *bulge* slightly at the Equator, which has the effect of further intensifying the variation, since a point on the surface of the Earth at the Equator is slightly further away from the Earth's center than a similar point at one of the Poles (and, hence, the true gravitational acceleration is slightly weaker in the former case).

Another consequence of centrifugal acceleration is that the apparent gravitational acceleration on the Earth's surface has a *horizontal* component aligned in the North/South direction. This horizontal component ensures that the apparent gravitational acceleration *does not* point directly towards the center of the Earth.

In other words, a plumb-line on the surface of the Earth does not point vertically downward, but is deflected slightly away from a true vertical in the North/South direction. The angular deviation from true vertical can easily be calculated from Eq. (8.16):

$$\theta_{dev} \simeq -\frac{\Omega^2\, R}{2\, g}\, \sin(2\,\lambda) \simeq -0.1^\circ\, \sin(2\,\lambda). \tag{8.18}$$

Here, a positive angle denotes a northward deflection, and *vice versa*. Thus, the deflection is *southward* in the *Northern Hemisphere* (*i.e.*, $\lambda > 0$) and *northwards* in the *Southern Hemisphere* (*i.e.*, $\lambda < 0$). The deflection is zero at the Poles and at the Equator, and reaches its maximum magnitude (which is very small) at middle latitudes.

8.4 The Coriolis force

We have now accounted for the first fictitious force, $-m\,\mathbf{\Omega}\times(\mathbf{\Omega}\times\mathbf{r})$, in Eq. (8.9). Let us now investigate the second, which takes the form $-2\,m\,\mathbf{\Omega}\times\mathbf{v}'$, and is called the *Coriolis force*. Obviously, this force only affects objects which are *moving* in the rotating reference frame.

Consider a particle of mass m free-falling under gravity in our rotating reference frame. As before, we define Cartesian axes in the rotating frame such that the z'-axis points vertically upward, and x'- and y'-axes are horizontal, with the x'-axis pointing directly northwards, and the y'-axis pointing directly westward. It follows, from Eq. (8.9), that the Cartesian equations of motion of the particle in the rotating reference frame take the form:

$$\ddot{x}' = 2\,\Omega\,\sin\lambda\,\dot{y}', \tag{8.19}$$

$$\ddot{y}' = -2\,\Omega\,\sin\lambda\,\dot{x}' + 2\,\Omega\,\cos\lambda\,\dot{z}', \tag{8.20}$$

$$\ddot{z}' = -g - 2\,\Omega\,\cos\lambda\,\dot{y}'. \tag{8.21}$$

Here, $\dot{}\equiv d/dt$, and g is the local acceleration due to gravity. In the above, we have neglected the centrifugal acceleration, for the sake of simplicity. This is reasonable, since the only effect of the centrifugal acceleration is to slightly modify the magnitude and direction of the local gravitational acceleration.

Consider a particle which is dropped (at $t = 0$) from rest a height h above the Earth's surface. To lowest order (*i.e.*, neglecting Ω), the particle's vertical motion satisfies

$$z' = h - \frac{g\,t^2}{2}. \tag{8.22}$$

Substituting this expression into Eqs. (8.19) and (8.20), and neglecting terms involving Ω^2, we obtain $x' \simeq 0$, and

$$y' \simeq -g\,\Omega\,\cos\lambda\,\frac{t^3}{3}. \tag{8.23}$$

In other words, the particle is deflected *eastward* (*i.e.*, in the negative y'-direction). Now, the particle hits the ground when $t \simeq \sqrt{2\,h/g}$. Hence, the net eastward deflection of the particle as strikes the ground is

$$d_{east} = \frac{\Omega}{3}\,\cos\lambda\,\left(\frac{8\,h^3}{g}\right)^{1/2}. \tag{8.24}$$

Note that this deflection is in the *same* direction as the Earth's rotation (*i.e.*, West to East), and is greatest at the Equator, and zero at the Poles. A particle dropped from a height of 100 m at the Equator is deflected by about 2.2 cm.

Consider a particle launched *horizontally* with some fairly large velocity

$$\mathbf{V} = V_0\,(\cos\theta, -\sin\theta, 0). \tag{8.25}$$

Here, θ is the *compass bearing* of the velocity vector (so North is 0°, East is 90°, *etc.*). Neglecting any vertical motion, Eqs. (8.19) and (8.20) yield

$$\dot{v}_{x'} \simeq -2\,\Omega\,V_0\,\sin\lambda\,\sin\theta, \tag{8.26}$$

$$\dot{v}'_y \simeq -2\,\Omega\,V_0\,\sin\lambda\,\cos\theta, \tag{8.27}$$

which can be integrated to give

$$v_{x'} \simeq V_0\,\cos\theta - 2\,\Omega\,V_0\,\sin\lambda\,\sin\theta\,t, \tag{8.28}$$

$$v_{y'} \simeq -V_0\,\sin\theta - 2\,\Omega\,V_0\,\sin\lambda\,\cos\theta\,t. \tag{8.29}$$

To lowest order in Ω, the above equations are equivalent to

$$v_{x'} \simeq V_0\,\cos(\theta + 2\,\Omega\,\sin\lambda\,t), \tag{8.30}$$

$$v_{y'} \simeq -V_0\,\sin(\theta + 2\,\Omega\,\sin\lambda\,t). \tag{8.31}$$

If follows that the Coriolis force causes the compass bearing of the particle's velocity vector to *rotate* steadily as time progresses. The rotation rate is

$$\frac{d\theta}{dt} \simeq 2\,\Omega\,\sin\lambda. \tag{8.32}$$

Hence, the rotation is *clockwise* (looking from above) in the *Northern Hemisphere*, and *anti-clockwise* in the *Southern Hemisphere*. The rotation rate is zero at the Equator, and greatest at the Poles.

The Coriolis force has a significant effect on terrestrial weather patterns. Near equatorial regions, the intense heating of the Earth's surface due to the Sun results in hot air rising. In the Northern Hemisphere, this causes cooler air to move in a southerly direction towards the Equator. The Coriolis force deflects this moving air in a clockwise sense (looking from above), resulting in the *trade winds*, which blow towards the *southwest*. In the Southern Hemisphere, the cooler air moves northwards, and is deflected by the Coriolis force in an anti-clockwise sense, resulting in trade winds which blow towards the *northwest*.

Furthermore, as air flows from high to low pressure regions, the Coriolis force deflects the air in a clockwise/anti-clockwise manner in the Northern/Southern Hemisphere, producing *cyclonic* rotation (see Fig. 38). It follows that cyclonic rotation is *anti-clockwise* in the *Northern Hemisphere*, and *clockwise* in the *Southern Hemisphere*. Thus, this is the direction of rotation of tropical storms (*e.g.*, hurricanes, typhoons) in each hemisphere.

8.5 The Foucault pendulum

Consider a pendulum consisting of a mass m suspended from a light cable of length l in such a manner that the pendulum is free to oscillate in any plane whose normal is parallel to the Earth's surface. The mass is subject to three forces: first, the force of gravity $m\,\mathbf{g}$, which is directed vertically downward (we are again ignoring centrifugal acceleration); second, the tension $\mathbf{T}$ in the cable, which is directly upward along the cable; and, third, the Coriolis force. It follows that the apparent equation of motion of the mass, in a frame of reference which co-rotates

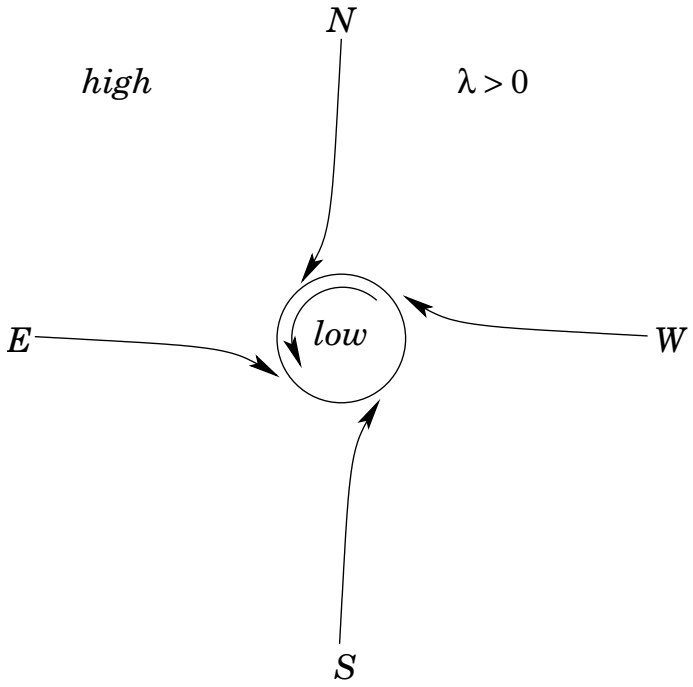

Figure 38:

with the Earth, is [see Eq. (8.9)]

$$m\,\ddot{\mathbf{r}}' = m\,\mathbf{g} + \mathbf{T} - 2\,m\,\mathbf{\Omega}\times\dot{\mathbf{r}}'. \tag{8.33}$$

Let us define our usual Cartesian coordinates (x', y', z'), and let the origin of our coordinate system correspond to the equilibrium position of the mass. If the pendulum is deflected by a *small* angle θ then it is easily seen that $x' \sim l\,\theta$, $y' \sim l\,\theta$, and $z' \sim l\,\theta^2$. In other words, the change in height of the mass, z', is negligible compared to its horizontal displacement. Hence, we can write $z' \simeq 0$, provided that $\theta \ll 1$. The tension $\mathbf{T}$ has the vertical component $T\,\cos\theta \simeq T$, and the horizontal component $\mathbf{T}_{hz} = -T\,\sin\theta\,\mathbf{r}'/r' \simeq -T\,\mathbf{r}'/l$, since $\sin\theta \simeq r'/l$ (see Fig. 39). Hence, the Cartesian equations of motion of the mass are written [*cf.*, Eqs. (8.19)–(8.21)]

$$\ddot{x}' = -\frac{T}{l\,m}\,x' + 2\,\Omega\,\sin\lambda\,\dot{y}', \tag{8.34}$$

$$\ddot{y}' = -\frac{T}{l\,m}\,y' - 2\,\Omega\,\sin\lambda\,\dot{x}', \tag{8.35}$$

$$0 = \frac{T}{m} - g - 2\,\Omega\,\cos\lambda\,\dot{y}'. \tag{8.36}$$

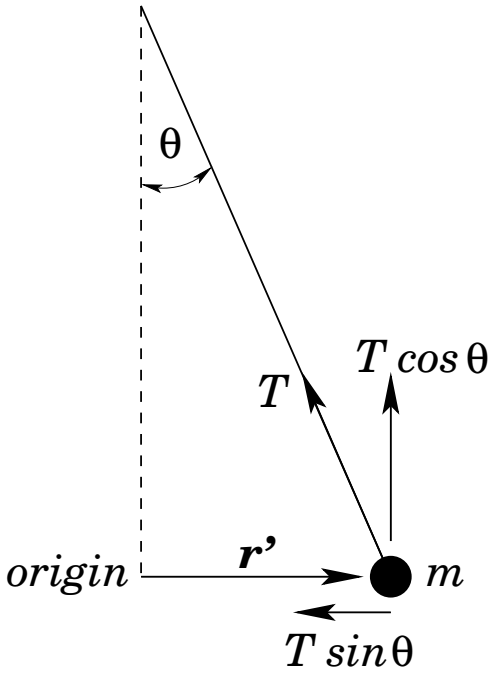

Figure 39:

To lowest order in Ω (*i.e.*, neglecting Ω), the final equation, which is just vertical force balance, yields $T \simeq m\,g$. Hence, Eqs. (8.34) and (8.35) reduce to

$$\ddot{x}' \simeq -\frac{g}{l}\,x' + 2\,\Omega\,\sin\lambda\,\dot{y}', \tag{8.37}$$

$$\ddot{y}' \simeq -\frac{g}{l}\,y' - 2\,\Omega\,\sin\lambda\,\dot{x}'. \tag{8.38}$$

Let

$$s = x' + \mathrm{i}\,y'. \tag{8.39}$$

Equations (8.37) and (8.38) can be combined to give a single complex equation for s:

$$\ddot{s} = -\frac{g}{l}\,s - \mathrm{i}\,2\,\Omega\,\sin\lambda\,\dot{s}. \tag{8.40}$$

Let us look for a sinusoidally oscillating solution of the form

$$s = s_0\,\mathrm{e}^{-\mathrm{i}\,\omega\,t}. \tag{8.41}$$

Here, ω is the (real) angular frequency of oscillation, and s_0 is an arbitrary complex

constant. Equations (8.40) and (8.41) yield the following quadratic equation for ω:

$$\omega^2 - 2\,\Omega\,\sin\lambda\,\omega - \frac{g}{l} = 0. \tag{8.42}$$

The solutions are approximately

$$\omega_\pm \simeq \Omega\,\sin\lambda \pm \sqrt{\frac{g}{l}}, \tag{8.43}$$

where we have neglected terms involving Ω^2. Hence, the general solution of (8.41) takes the form

$$s = s_+\,\mathrm{e}^{-\mathrm{i}\,\omega_+\,t} + s_-\,\mathrm{e}^{-\mathrm{i}\,\omega_-\,t}, \tag{8.44}$$

where s_+ and s_- are two arbitrary complex constants.

Making the specific choice $s_+ = s_- = a/2$, where a is real, the above solution reduces to

$$s = a\,\mathrm{e}^{-\mathrm{i}\,\Omega\,\sin\lambda\,t}\,\cos\left(\sqrt{\frac{g}{l}}\,t\right). \tag{8.45}$$

Now, it is clear from Eq. (8.39) that x' and y' are the real and imaginary parts of s, respectively. Thus, it follows from the above that

$$x' = a\,\cos(\Omega\,\sin\lambda\,t)\,\cos\left(\sqrt{\frac{g}{l}}\,t\right), \tag{8.46}$$

$$y' = -a\,\sin(\Omega\,\sin\lambda\,t)\,\cos\left(\sqrt{\frac{g}{l}}\,t\right). \tag{8.47}$$

These equations describe sinusoidal oscillations, in a plane whose normal is parallel to the Earth's surface, at the standard pendulum frequency $\sqrt{g/l}$. The Coriolis force, however, causes the plane of oscillation to slowly *precess* at the angular frequency $\Omega\,\sin\lambda$. The period of the precession is

$$T = \frac{2\,\pi}{\Omega\,\sin\lambda} = \frac{24}{\sin\lambda}\ \text{hrs}. \tag{8.48}$$

For example, according to the above equations, the pendulum oscillates in the x'-direction (*i.e.*, North–South) at $t \simeq 0$, in the y'-direction (*i.e.*, East–West) at $t \simeq T/4$, in the x'-direction again at $t \simeq T/2$, *etc.* The precession is *clockwise* (looking from above) in the *Northern Hemisphere*, and *anti-clockwise* in the *Southern Hemisphere*.

The precession of the plane of oscillation of a pendulum, due to the Coriolis force, is used in many museums and observatories to demonstrate that the Earth is rotating. This method of making the Earth's rotation manifest was first devised by Foucault in 1851.

9 Rigid body motion

9.1 Introduction

In this section, we shall investigate the motion of rigid bodies in three dimensions.

9.2 Fundamental equations

We can think of a rigid body as a collection of a large number of small mass elements which all maintain a fixed spatial relationship with respect to one another. Let there be N elements, and let the ith element be of mass m_i, and instantaneous position vector $\mathbf{r}_i$. The equation of motion of the ith element is written

$$m_i \frac{d^2\mathbf{r}_i}{dt^2} = \sum_{j=1,N}^{j\neq i} \mathbf{f}_{ij} + \mathbf{F}_i. \tag{9.1}$$

Here, $\mathbf{f}_{ij}$ is the internal force exerted on the ith element by the jth element, and $\mathbf{F}_i$ the external force acting on the ith element. The internal forces $\mathbf{f}_{ij}$ represent the stresses which develop within the body in order to ensure that its various elements maintain a constant spatial relationship with respect to one another. Of course, $\mathbf{f}_{ij} = -\mathbf{f}_{ji}$, by Newton's third law. The external forces represent forces which originate outside the body.

Generalizing the analysis contained in Sect. 3.5, we can sum Eq. (9.1) over all mass elements to obtain

$$M \frac{d^2\mathbf{r}_{cm}}{dt^2} = \mathbf{F}. \tag{9.2}$$

Here, $M = \sum_{i=1,N} m_i$ is the total mass, $\mathbf{r}_{cm}$ the position vector of the center of mass [see Eq. (3.27)], and $\mathbf{F} = \sum_{i=1,N} \mathbf{F}_i$ the total external force. It can be seen that the center of mass of a rigid body moves under the action of the external forces as a point particle whose mass is identical with that of the body.

Again generalizing the analysis of Sect. 3.5, we can sum $\mathbf{r}_i\times$ Eq. (9.1) over all mass elements to obtain

$$\frac{d\mathbf{L}}{dt} = \mathbf{T}. \tag{9.3}$$

Here, $\mathbf{L} = \sum_{i=1,N} m_i\, \mathbf{r}_i \times d\mathbf{r}_i/dt$ is the total angular momentum of the body (about the origin), and $\mathbf{T} = \sum_{i=1,N} \mathbf{r}_i \times \mathbf{F}_i$ the total external torque (about the origin). Note that the above equation is only valid if the internal forces are *central* in nature. However, this is not a particularly onerous constraint. Equation (9.3) describes how the angular momentum of a rigid body evolves in time under the action of the external torques.

In the following, we shall only consider the *rotational* motion of rigid bodies, since their translational motion is similar to that of point particles [see Eq. (9.2)], and, therefore, fairly straightforward in nature.

9.3 The moment of inertia tensor

Consider a rigid body rotating with fixed angular velocity $\boldsymbol{\omega}$ about an axis which passes through the origin (see Fig. 40). Let $\mathbf{r}_i$ be the position vector of the ith mass element, whose mass is m_i. We expect this position vector to *precess* about the axis of rotation (which is parallel to $\boldsymbol{\omega}$) with angular velocity ω. It, therefore, follows from Eq. (2.41) that

$$\frac{d\mathbf{r}_i}{dt} = \boldsymbol{\omega} \times \mathbf{r}_i. \tag{9.4}$$

Thus, the above equation specifies the velocity, $\mathbf{v}_i = d\mathbf{r}_i/dt$, of each mass element as the body rotates with fixed angular velocity $\boldsymbol{\omega}$ about an axis passing through the origin.

The total angular momentum of the body (about the origin) is written

$$\mathbf{L} = \sum_{i=1,N} m_i\, \mathbf{r}_i \times \frac{d\mathbf{r}_i}{dt} = \sum_{i=1,N} m_i\, \mathbf{r}_i \times (\boldsymbol{\omega} \times \mathbf{r}_i) = \sum_{i=1,N} m_i \left[r_i^2\, \boldsymbol{\omega} - (\mathbf{r}_i \cdot \boldsymbol{\omega})\, \mathbf{r}_i \right], \tag{9.5}$$

where use has been made of Eq. (9.4), and some standard vector identities. The above formula can be written as a matrix equation of the form

$$\begin{pmatrix} L_x \\ L_y \\ L_z \end{pmatrix} = \begin{pmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{pmatrix} \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix}, \tag{9.6}$$

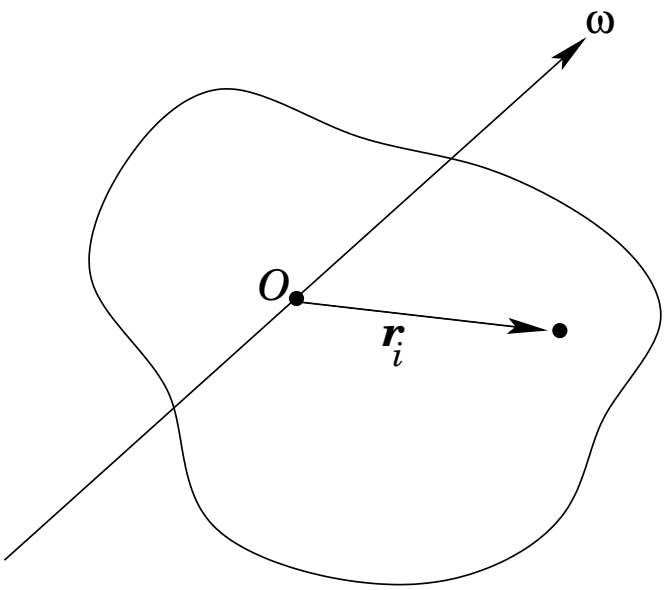

Figure 40:

where

$$I_{xx} = \sum_{i=1,N} (y_i^2 + z_i^2)\, m_i = \int (y^2 + z^2)\, dm, \tag{9.7}$$

$$I_{yy} = \sum_{i=1,N} (x_i^2 + z_i^2)\, m_i = \int (x^2 + z^2)\, dm, \tag{9.8}$$

$$I_{zz} = \sum_{i=1,N} (x_i^2 + y_i^2)\, m_i = \int (x^2 + y^2)\, dm, \tag{9.9}$$

$$I_{xy} = I_{yx} = -\sum_{i=1,N} x_i\, y_i\, m_i = -\int x\, y\, dm, \tag{9.10}$$

$$I_{yz} = I_{zy} = -\sum_{i=1,N} y_i\, z_i\, m_i = -\int y\, z\, dm, \tag{9.11}$$

$$I_{xz} = I_{zx} = -\sum_{i=1,N} x_i\, z_i\, m_i = -\int x\, z\, dm. \tag{9.12}$$

Here, I_{xx} is called the *moment of inertia* about the x-axis, I_{yy} the moment of inertia about the y-axis, I_{xy} the *xy product of inertia*, I_{yz} the yz product of inertia, *etc.* The matrix of the I_{ij} values is known as the *moment of inertia tensor*.[1] Note that each component of the moment of inertia tensor can be written as either a sum

[1] A tensor is the two-dimensional generalization of a vector. However, for our purposes, we can simply think of a tensor as another name for a matrix.

over separate mass elements, or an integral over infinitesimal mass elements. In the integrals, $dm = \rho\, dV$, where ρ is the mass density, and dV a volume element. Equation (9.6) can be written more succinctly as

$$\mathbf{L} = \tilde{\mathbf{I}}\,\boldsymbol{\omega}. \tag{9.13}$$

Here, it is understood that $\mathbf{L}$ and $\boldsymbol{\omega}$ are both *column vectors*, and $\tilde{\mathbf{I}}$ is the *matrix* of the I_{ij} values. Note that $\tilde{\mathbf{I}}$ is a *real symmetric* matrix: *i.e.*, $I_{ij}^* = I_{ij}$ and $I_{ji} = I_{ij}$.

In general, the angular momentum vector, $\mathbf{L}$, obtained from Eq. (9.13), points in a different direction to the angular velocity vector, $\boldsymbol{\omega}$. In other words, $\mathbf{L}$ is generally *not parallel* to $\boldsymbol{\omega}$.

Finally, although the above results were obtained assuming a fixed angular velocity, they remain valid at each instant in time even if the angular velocity varies.

9.4 Rotational kinetic energy

The instantaneous rotational kinetic energy of a rotating rigid body is written

$$K = \frac{1}{2} \sum_{i=1,N} m_i \left(\frac{d\mathbf{r}_i}{dt}\right)^2. \tag{9.14}$$

Making use of Eq. (9.4), and some vector identities, the kinetic energy takes the form

$$K = \frac{1}{2} \sum_{i=1,N} m_i\, (\boldsymbol{\omega} \times \mathbf{r}_i) \cdot (\boldsymbol{\omega} \times \mathbf{r}_i) = \frac{1}{2}\, \boldsymbol{\omega} \cdot \sum_{i=1,N} m_i\, \mathbf{r}_i \times (\boldsymbol{\omega} \times \mathbf{r}_i). \tag{9.15}$$

Hence, it follows from (9.5) that

$$K = \frac{1}{2}\, \boldsymbol{\omega} \cdot \mathbf{L}. \tag{9.16}$$

Making use of Eq. (9.13), we can also write

$$K = \frac{1}{2}\, \boldsymbol{\omega}^T\, \tilde{\mathbf{I}}\, \boldsymbol{\omega}. \tag{9.17}$$

Here, $\boldsymbol{\omega}^T$ is the *row vector* of the Cartesian components ω_x, ω_y, ω_z, which is, of course, the transpose (denoted ^{T}) of the column vector $\boldsymbol{\omega}$. When written in component form, the above equation yields

$$K = \frac{1}{2}\left(I_{xx}\,\omega_x^2 + I_{yy}\,\omega_y^2 + I_{zz}\,\omega_z^2 + 2\,I_{xy}\,\omega_x\,\omega_y + 2\,I_{yz}\,\omega_y\,\omega_z + 2\,I_{xz}\,\omega_x\,\omega_z\right). \quad (9.18)$$

9.5 Matrix theory

It is time to review a little matrix theory. Suppose that $\mathbf{A}$ is a *real symmetric* matrix of dimension n. If follows that $\mathbf{A}^* = \mathbf{A}$ and $\mathbf{A}^T = \mathbf{A}$, where * denotes a complex conjugate, and ^{T} denotes a transpose. Consider the matrix equation

$$\mathbf{A}\,\mathbf{x} = \lambda\,\mathbf{x}. \quad (9.19)$$

Any column vector $\mathbf{x}$ which satisfies the above equation is called an *eigenvector* of $\mathbf{A}$. Likewise, the associated number λ is called an *eigenvalue* of $\mathbf{A}$. Let us investigate the properties of the eigenvectors and eigenvalues of a real symmetric matrix.

Equation (9.19) can be rearranged to give

$$(\mathbf{A} - \lambda\,\mathbf{1})\,\mathbf{x} = \mathbf{0}, \quad (9.20)$$

where $\mathbf{1}$ is the unit matrix. The above matrix equation is essentially a set of n homogeneous simultaneous algebraic equations for the n components of $\mathbf{x}$. A well-known property of such a set of equations is that it only has a non-trivial solution when the determinant of the associated matrix is set to zero. Hence, a necessary condition for the above set of equations to have a non-trivial solution is that

$$|\mathbf{A} - \lambda\,\mathbf{1}| = 0. \quad (9.21)$$

The above formula is essentially an nth-order *polynomial* equation for λ. We know that such an equation has n (possibly complex) roots. Hence, we conclude that there are n eigenvalues, and n associated eigenvectors, of the n-dimensional matrix $\mathbf{A}$.

Let us now demonstrate that the n eigenvalues and eigenvectors of the real symmetric matrix $\mathbf{A}$ are all *real*. We have

$$\mathbf{A}\,\mathbf{x}_i = \lambda_i\,\mathbf{x}_i, \tag{9.22}$$

and, taking the transpose and complex conjugate,

$$\mathbf{x}_i^{*T}\,\mathbf{A} = \lambda_i^*\,\mathbf{x}_i^{*T}, \tag{9.23}$$

where $\mathbf{x}_i$ and λ_i are the ith eigenvector and eigenvalue of $\mathbf{A}$ respectively. Left multiplying Eq. (9.22) by $\mathbf{x}_i^{*T}$, we obtain

$$\mathbf{x}_i^{*T}\mathbf{A}\,\mathbf{x}_i = \lambda_i\,\mathbf{x}_i^{*T}\mathbf{x}_i. \tag{9.24}$$

Likewise, right multiplying (9.23) by $\mathbf{x}_i$, we get

$$\mathbf{x}_i^{*T}\,\mathbf{A}\,\mathbf{x}_i = \lambda_i^*\,\mathbf{x}_i^{*T}\mathbf{x}_i. \tag{9.25}$$

The difference of the previous two equations yields

$$(\lambda_i - \lambda_i^*)\,\mathbf{x}_i^{*T}\mathbf{x}_i = 0. \tag{9.26}$$

It follows that $\lambda_i = \lambda_i^*$, since $\mathbf{x}_i^{*T}\mathbf{x}_i$ (which is $\mathbf{x}_i^* \cdot \mathbf{x}_i$ in vector notation) is positive definite. Hence, λ_i is real. It immediately follows that $\mathbf{x}_i$ is real.

Next, let us show that two eigenvectors corresponding to two *different* eigenvalues are *mutually orthogonal*. Let

$$\mathbf{A}\,\mathbf{x}_i = \lambda_i\,\mathbf{x}_i, \tag{9.27}$$

$$\mathbf{A}\,\mathbf{x}_j = \lambda_j\,\mathbf{x}_j, \tag{9.28}$$

where $\lambda_i \neq \lambda_j$. Taking the transpose of the first equation and right multiplying by $\mathbf{x}_j$, and left multiplying the second equation by $\mathbf{x}_i^T$, we obtain

$$\mathbf{x}_i^T\,\mathbf{A}\,\mathbf{x}_j = \lambda_i\,\mathbf{x}_i^T\mathbf{x}_j, \tag{9.29}$$

$$\mathbf{x}_i^T\,\mathbf{A}\,\mathbf{x}_j = \lambda_j\,\mathbf{x}_i^T\mathbf{x}_j. \tag{9.30}$$

Taking the difference of the above two equations, we get

$$(\lambda_i - \lambda_j)\,\mathbf{x}_i^T\mathbf{x}_j = 0. \tag{9.31}$$

Since, by hypothesis, $\lambda_i \neq \lambda_j$, it follows that $\mathbf{x}_i^T \mathbf{x}_j = 0$. In vector notation, this is the same as $\mathbf{x}_i \cdot \mathbf{x}_j = 0$. Hence, the eigenvectors $\mathbf{x}_i$ and $\mathbf{x}_j$ are mutually orthogonal.

Suppose that $\lambda_i = \lambda_j = \lambda$. In this case, we cannot conclude that $\mathbf{x}_i^T \mathbf{x}_j = 0$ by the above argument. However, it is easily seen that any linear combination of $\mathbf{x}_i$ and $\mathbf{x}_j$ is an eigenvector of $\mathbf{A}$ with eigenvalue λ. Hence, it is possible to define two new eigenvectors of $\mathbf{A}$, with the eigenvalue λ, which are mutually orthogonal. For instance,

$$\mathbf{x}_i' = \mathbf{x}_i, \tag{9.32}$$

$$\mathbf{x}_j' = \mathbf{x}_j - \left(\frac{\mathbf{x}_i^T \mathbf{x}_j}{\mathbf{x}_i^T \mathbf{x}_i}\right) \mathbf{x}_i. \tag{9.33}$$

It should be clear that this argument can be generalized to deal with any number of eigenvalues which take the same value.

In conclusion, a real symmetric, n-dimensional matrix possesses n *real* eigenvalues, with n associated *real* eigenvectors, which are, or can be chosen to be, *mutually orthogonal*.

9.6 The principal axes of rotation

We have seen that the moment of inertia tensor, $\tilde{\mathbf{I}}$, defined in Sect. 9.3, takes the form of a real symmetric, three-dimensional matrix. It therefore follows, from the matrix theory which we have just reviewed, that the moment of inertia tensor possesses *three mutually orthogonal eigenvectors* which are associated with *three real eigenvalues*. Let the ith eigenvector (which can be normalized to be a unit vector) be denoted $\hat{\boldsymbol{\omega}}_i$, and the ith eigenvalue λ_i. It then follows that

$$\tilde{\mathbf{I}}\, \hat{\boldsymbol{\omega}}_i = \lambda_i\, \hat{\boldsymbol{\omega}}_i, \tag{9.34}$$

for $i = 1, 3$.

The directions of the three mutually orthogonal unit vectors $\hat{\boldsymbol{\omega}}_i$ define the three so-called *principal axes of rotation* of the rigid body under investigation. These axes are special because when the body rotates about one of them (*i.e.*, when $\boldsymbol{\omega}$ is

parallel to one of them) the angular momentum vector $\mathbf{L}$ becomes *parallel* to the angular velocity vector $\boldsymbol{\omega}$. This can be seen from a comparison of Eq. (9.13) and Eq. (9.34).

Suppose that we reorient our Cartesian coordinate axes so the they coincide with the mutually orthogonal principal axes of rotation. In this new reference frame, the eigenvectors of $\tilde{\mathbf{I}}$ are the unit vectors, $\mathbf{e}_x$, $\mathbf{e}_y$, and $\mathbf{e}_z$, and the eigenvalues are the moments of inertia about these axes, I_{xx}, I_{yy}, and I_{zz}, respectively. These latter quantities are referred to as the *principal moments of inertia*. Note that the products of inertia are all *zero* in the new reference frame. Hence, in this frame, the moment of inertia tensor takes the form of a *diagonal* matrix: *i.e.*,

$$\tilde{\mathbf{I}} = \begin{pmatrix} I_{xx} & 0 & 0 \\ 0 & I_{yy} & 0 \\ 0 & 0 & I_{zz} \end{pmatrix}. \tag{9.35}$$

Incidentally, it is easy to verify that $\mathbf{e}_x$, $\mathbf{e}_y$, and $\mathbf{e}_z$ are indeed the eigenvectors of the above matrix, with the eigenvalues I_{xx}, I_{yy}, and I_{zz}, respectively, and that $\mathbf{L} = \tilde{\mathbf{I}}\,\boldsymbol{\omega}$ is indeed parallel to $\boldsymbol{\omega}$ whenever $\boldsymbol{\omega}$ is directed along $\mathbf{e}_x$, $\mathbf{e}_y$, or $\mathbf{e}_z$.

When expressed in our new coordinate system, Eq. (9.13) yields

$$\mathbf{L} = (I_{xx}\,\omega_x,\, I_{yy}\,\omega_y, I_{zz}\,\omega_z)\,, \tag{9.36}$$

whereas Eq. (9.18) reduces to

$$K = \frac{1}{2}\left(I_{xx}\,\omega_x^2 + I_{yy}\,\omega_y^2 + I_{zz}\,\omega_z^2\right). \tag{9.37}$$

In conclusion, there are many great simplifications to be had by choosing a coordinate system whose axes coincide with the principal axes of rotation of the rigid body under investigation. But how do we determine the directions of the principal axes in practice?

Well, in general, we have to solve the eigenvalue equation

$$\tilde{\mathbf{I}}\,\hat{\boldsymbol{\omega}} = \lambda\,\hat{\boldsymbol{\omega}}, \tag{9.38}$$

or

$$\left(\begin{array}{ccc} I_{xx}-\lambda & I_{xy} & I_{xz} \\ I_{yx} & I_{yy}-\lambda & I_{yz} \\ I_{zx} & I_{zy} & I_{zz}-\lambda \end{array}\right)\left(\begin{array}{c} \cos\alpha \\ \cos\beta \\ \cos\gamma \end{array}\right)=\left(\begin{array}{c} 0 \\ 0 \\ 0 \end{array}\right), \tag{9.39}$$

where $\hat{\boldsymbol{\omega}} = (\cos\alpha, \cos\beta, \cos\gamma)$, and $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$. Here, α is the angle the unit eigenvector subtends with the x-axis, β the angle it subtends with the y-axis, and γ the angle it subtends with the z-axis. Unfortunately, the analytic solution of the above matrix equation is generally quite difficult.

Fortunately, however, in many instances the rigid body under investigation possesses some kind of symmetry, so that at least one principal axis can be found by inspection. In this case, the other two principal axes can be determined as follows.

Suppose that the z-axis is known to be a principal axes (at the origin) in some coordinate system. It follows that the two products of inertia I_{xz} and I_{yz} are zero [otherwise, (0, 0, 1) would not be an eigenvector in Eq. (9.39)]. The other two principal axes must lie in the x-y plane: *i.e.*, $\cos\gamma = 0$. It then follows that $\cos\beta = \sin\alpha$, since $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$. The first two rows in the matrix equation (9.39) thus reduce to

$$(I_{xx}-\lambda)\cos\alpha + I_{xy}\sin\alpha = 0, \tag{9.40}$$

$$I_{xy}\cos\alpha + (I_{yy}-\lambda)\sin\alpha = 0. \tag{9.41}$$

Eliminating λ between the above two equations, we obtain

$$I_{xy}(1-\tan^2\alpha) = (I_{xx}-I_{yy})\tan\alpha. \tag{9.42}$$

But, $\tan(2\alpha) \equiv 2\tan\alpha/(1-\tan^2\alpha)$. Hence, Eq. (9.42) yields

$$\tan(2\alpha) = \frac{2\,I_{xy}}{I_{xx}-I_{yy}}. \tag{9.43}$$

There are two values of α, lying between $-\pi/2$ and $\pi/2$, which satisfy the above equation. These specify the angles, α, which the two mutually orthogonal principal axes in the x-y plane make with the x-axis. Hence, we have now determined the directions of all three principal axes. Incidentally, once we have determined the orientation angle, α, of a principal axis, we can substitute back into Eq. (9.40) to obtain the corresponding principal moment of inertia, λ.

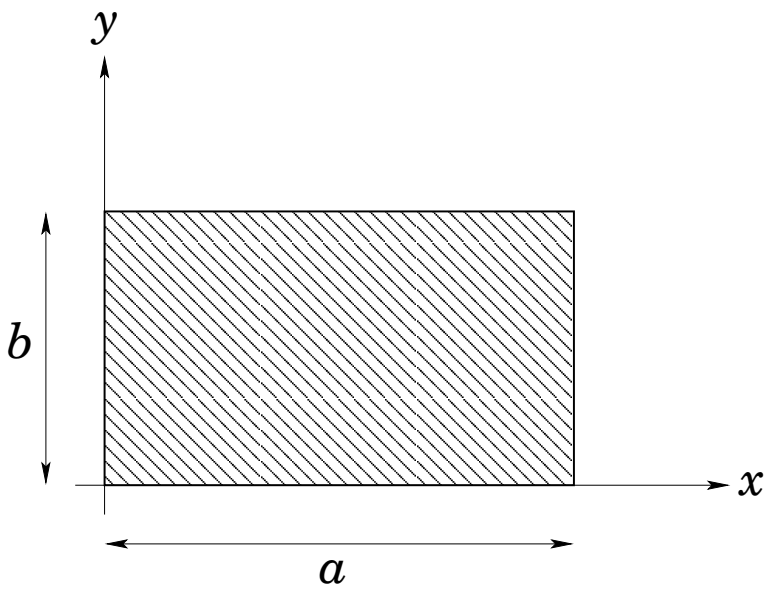

Figure 41:

As an example, consider a uniform rectangular lamina of mass m and sides a and b which lies in the x-y plane, as shown in Fig. 41. Suppose that the axis of rotation passes through the origin (*i.e.*, through a corner of the lamina). Since $z = 0$ throughout the lamina, it follows from Eqs. (9.11) and (9.12) that $I_{xz} = I_{yz} = 0$. Hence, the z-axis is a principal axis. After some straightforward integration, Eqs. (9.7)–(9.10) yield

$$I_{xx} = \frac{1}{3} m\, b^2, \tag{9.44}$$

$$I_{yy} = \frac{1}{3} m\, a^2, \tag{9.45}$$

$$I_{xy} = -\frac{1}{4} m\, a\, b. \tag{9.46}$$

Thus, it follows from Eq. (9.43) that

$$\alpha = \frac{1}{2} \tan^{-1}\left(\frac{3}{2}\frac{a\, b}{a^2 - b^2}\right). \tag{9.47}$$

The above equation specifies the orientation of the two principal axes which lie in the x-y plane. For the special case where $a = b$, we get $\alpha = \pi/4,\, 3\pi/4$: *i.e.*, the two in-plane principal axes of a square lamina (at a corner) are parallel to the two diagonals of the lamina.

9.7 Euler's equations

The fundamental equation of motion of a rotating body [see Eq. (9.3)],

$$\mathbf{T} = \frac{d\mathbf{L}}{dt}, \tag{9.48}$$

is only valid in an *inertial* frame. However, we have seen that $\mathbf{L}$ is most simply expressed in a frame of reference whose axes are aligned along the principal axes of rotation of the body. Such a frame of reference *rotates* with the body, and is, therefore, *non-inertial*. Thus, it is helpful to define *two* Cartesian coordinate systems, with the same origins. The first, with coordinates x, y, z, is a fixed inertial frame—let us denote this the *fixed frame*. The second, with coordinates x', y', z', co-rotates with the body in such a manner that the x'-, y'-, and z'-axes are always pointing along its principal axes of rotation—we shall refer to this as the *body frame*. Since the body frame co-rotates with the body, its instantaneous angular velocity is the same as that of the body. Hence, it follows from the analysis in Sect. 8.2 that

$$\frac{d\mathbf{L}}{dt} = \frac{d\mathbf{L}}{dt'} + \boldsymbol{\omega} \times \mathbf{L}. \tag{9.49}$$

Here, d/dt is the time derivative in the fixed frame, and d/dt' the time derivative in the body frame. Combining Eqs. (9.48) and (9.49), we obtain

$$\mathbf{T} = \frac{d\mathbf{L}}{dt'} + \boldsymbol{\omega} \times \mathbf{L}. \tag{9.50}$$

Now, in the body frame let $\mathbf{T} = (T_{x'}, T_{y'}, T_{z'})$ and $\boldsymbol{\omega} = (\omega_{x'}, \omega_{y'}, \omega_{z'})$. It follows that $\mathbf{L} = (I_{x'x'}\,\omega_{x'}, I_{y'y'}\,\omega_{y'}, I_{z'z'}\,\omega_{z'})$, where $I_{x'x'}$, $I_{y'y'}$ and $I_{z'z'}$ are the principal moments of inertia. Hence, in the body frame, the components of Eq. (9.50) yield

$$T_{x'} = I_{x'x'}\,\dot{\omega}_{x'} - (I_{y'y'} - I_{z'z'})\,\omega_{y'}\,\omega_{z'}, \tag{9.51}$$

$$T_{y'} = I_{y'y'}\,\dot{\omega}_{y'} - (I_{z'z'} - I_{x'x'})\,\omega_{z'}\,\omega_{x'}, \tag{9.52}$$

$$T_{z'} = I_{z'z'}\,\dot{\omega}_{z'} - (I_{x'x'} - I_{y'y'})\,\omega_{x'}\,\omega_{y'}, \tag{9.53}$$

where $\dot{} = d/dt$. Here, we have made use of the fact that the moments of inertia of a rigid body are *constant* in time in the co-rotating body frame. The above equations are known as *Euler's equations*.

Consider a rigid body which is constrained to rotate about a fixed axis with *constant* angular velocity. It follows that $\dot{\omega}_{x'} = \dot{\omega}_{y'} = \dot{\omega}_{z'} = 0$. Hence, Euler's equations, (9.51)–(9.53), reduce to

$$T_{x'} = -(I_{y'y'} - I_{z'z'})\,\omega_{y'}\,\omega_{z'}, \tag{9.54}$$

$$T_{y'} = -(I_{z'z'} - I_{x'x'})\,\omega_{z'}\,\omega_{x'}, \tag{9.55}$$

$$T_{z'} = -(I_{x'x'} - I_{y'y'})\,\omega_{x'}\,\omega_{y'}. \tag{9.56}$$

These equations specify the components of the steady (in the body frame) torque exerted on the body by the constraining supports. The steady (in the body frame) angular momentum is written

$$\mathbf{L} = (I_{x'x'}\,\omega_{x'},\, I_{y'y'}\,\omega_{y'},\, I_{z'z'}\,\omega_{z'}). \tag{9.57}$$

It is easily demonstrated that $\mathbf{T} = \boldsymbol{\omega} \times \mathbf{L}$. Hence, the torque is perpendicular to both the angular velocity and the angular momentum vectors. Note that if the axis of rotation is a principal axis then two of the three components of $\boldsymbol{\omega}$ are zero (in the body frame). It follows from Eqs. (9.54)–(9.56) that all three components of the torque are zero. In other words, *zero* external torque is required to make the body rotate steadily about a *principal axis*.

Suppose that the body is *freely rotating*: *i.e.*, there are no external torques. Furthermore, let the body be *rotationally symmetric* about the z'-axis. It follows that $I_{x'x'} = I_{y'y'} = I_\perp$. Likewise, we can write $I_{z'z'} = I_\parallel$. In general, however, $I_\perp \neq I_\parallel$. Thus, Euler's equations yield

$$I_\perp \frac{d\omega_{x'}}{dt} + (I_\parallel - I_\perp)\,\omega_{z'}\,\omega_{y'} = 0, \tag{9.58}$$

$$I_\perp \frac{d\omega_{y'}}{dt} - (I_\parallel - I_\perp)\,\omega_{z'}\,\omega_{x'} = 0, \tag{9.59}$$

$$\frac{d\omega_{z'}}{dt} = 0. \tag{9.60}$$

Clearly, $\omega_{z'}$ is a constant of the motion. Equation (9.58) and (9.59) can be written

$$\frac{d\omega_{x'}}{dt} + \Omega\,\omega_{y'} = 0, \tag{9.61}$$

$$\frac{d\omega_{y'}}{dt} - \Omega\,\omega_{x'} = 0, \tag{9.62}$$

where $\Omega = (I_\parallel/I_\perp - 1)\,\omega_{z'}$. As is easily demonstrated, the solution to the above equations is

$$\omega_{x'} = \omega_\perp \cos(\Omega\, t), \tag{9.63}$$

$$\omega_{y'} = \omega_\perp \sin(\Omega\, t), \tag{9.64}$$

where $\omega_\perp$ is a constant. Thus, the projection of the angular velocity vector onto the x'-y' plane has the fixed length $\omega_\perp$, and rotates steadily about the z'-axis with angular velocity Ω. It follows that the length of the angular velocity vector, $\omega = (\omega_{x'}^2 + \omega_{y'}^2 + \omega_{z'}^2)^{1/2}$, is a constant of the motion. Clearly, the angular velocity vector makes some constant angle, α, with the z'-axis, which implies that $\omega_{z'} = \omega\cos\alpha$ and $\omega_\perp = \omega\sin\alpha$. Hence, the components of the angular velocity vector are

$$\omega_{x'} = \omega\sin\alpha\cos(\Omega\, t), \tag{9.65}$$

$$\omega_{y'} = \omega\sin\alpha\sin(\Omega\, t), \tag{9.66}$$

$$\omega_{z'} = \omega\cos\alpha, \tag{9.67}$$

where

$$\Omega = \omega\cos\alpha\left(\frac{I_\parallel}{I_\perp} - 1\right). \tag{9.68}$$

We conclude that, in the body frame, the angular velocity vector *precesses* about the symmetry axis (*i.e.*, the z'-axis) with the angular frequency Ω. Now, the components of the angular momentum vector are

$$L_{x'} = I_\perp\,\omega\sin\alpha\cos(\Omega\, t), \tag{9.69}$$

$$L_{y'} = I_\perp\,\omega\sin\alpha\sin(\Omega\, t), \tag{9.70}$$

$$L_{z'} = I_\parallel\,\omega\cos\alpha. \tag{9.71}$$

Thus, in the body frame, the angular momentum vector is also of constant length, and precesses about the symmetry axis with the angular frequency Ω. Furthermore, the angular momentum vector makes a constant angle θ with the symmetry axis, where

$$\tan\theta = \frac{I_\perp}{I_\parallel}\tan\alpha. \tag{9.72}$$

Note that the angular momentum vector, the angular velocity vector, and the symmetry axis all lie in the *same plane*: *i.e.*, $\mathbf{e}_{z'}\cdot\mathbf{L}\times\boldsymbol{\omega} = 0$, as can easily be

verified. Moreover, the angular momentum vector lies between the angular velocity vector and the symmetry axis (*i.e.*, $\theta < \alpha$) for a flattened (or oblate) body (*i.e.*, $I_\perp < I_\parallel$), whereas the angular velocity vector lies between the angular momentum vector and the symmetry axis (*i.e.*, $\theta > \alpha$) for an elongated (or prolate) body (*i.e.*, $I_\perp > I_\parallel$).

9.8 Eulerian angles

We have seen how we can solve Euler's equations to determine the properties of a rotating body in the co-rotating *body frame*. Let us now investigate how we can determine the same properties in the inertial *fixed frame*.

The fixed frame and the body frame share the same origin. Hence, we can transform from one to the other by means of an appropriate *rotation* of our vector space. In general, if we restrict ourselves to rotations about one of the Cartesian coordinate axes, *three* successive rotations are required to transform the fixed frame into the body frame. There are, in fact, many different ways to combined three successive rotations in order to achieve this. In the following, we shall describe the most widely used method, which is due to Euler.

We start in the fixed frame, which has coordinates x, y, z, and unit vectors $\mathbf{e}_x$, $\mathbf{e}_y$, $\mathbf{e}_z$. Our first rotation is counterclockwise (looking down the axis) through an angle ϕ about the z-axis. The new frame has coordinates x'', y'', z'', and unit vectors $\mathbf{e}_{x''}$, $\mathbf{e}_{y''}$, $\mathbf{e}_{z''}$. According to Eqs. (2.7)–(2.9), the transformation of coordinates can be represented as follows:

$$\begin{pmatrix} x'' \\ y'' \\ z'' \end{pmatrix} = \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}. \tag{9.73}$$

The angular velocity vector associated with ϕ has the magnitude $\dot{\phi}$, and is directed along $\mathbf{e}_z$ (*i.e.*, along the axis of rotation). Hence, we can write

$$\boldsymbol{\omega}_\phi = \dot{\phi}\,\mathbf{e}_z. \tag{9.74}$$

Clearly, $\dot{\phi}$ is the precession rate about the $\mathbf{e}_z$ axis, as seen in the fixed frame.

The second rotation is counterclockwise (looking down the axis) through an angle θ about the x''-axis. The new frame has coordinates x''', y''', z''', and unit vectors $\mathbf{e}_{x'''}$, $\mathbf{e}_{y'''}$, $\mathbf{e}_{z'''}$. By analogy with Eq. (9.73), the transformation of coordinates can be represented as follows:

$$\begin{pmatrix} x''' \\ y''' \\ z''' \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x'' \\ y'' \\ z'' \end{pmatrix}. \tag{9.75}$$

The angular velocity vector associated with θ has the magnitude $\dot{\theta}$, and is directed along $\mathbf{e}_{x''}$ (*i.e.*, along the axis of rotation). Hence, we can write

$$\boldsymbol{\omega}_\theta = \dot{\theta}\,\mathbf{e}_{x''}. \tag{9.76}$$

The third rotation is counterclockwise (looking down the axis) through an angle ψ about the z'''-axis. The new frame is the body frame, which has coordinates x', y', z', and unit vectors $\mathbf{e}_{x'}$, $\mathbf{e}_{y'}$, $\mathbf{e}_{z'}$. The transformation of coordinates can be represented as follows:

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x''' \\ y''' \\ z''' \end{pmatrix}. \tag{9.77}$$

The angular velocity vector associated with ψ has the magnitude $\dot{\psi}$, and is directed along $\mathbf{e}_{z''}$ (*i.e.*, along the axis of rotation). Note that $\mathbf{e}_{z'''} = \mathbf{e}_{z'}$, since the third rotation is about $\mathbf{e}_{z'''}$. Hence, we can write

$$\boldsymbol{\omega}_\psi = \dot{\psi}\,\mathbf{e}_{z'}. \tag{9.78}$$

Clearly, $\dot{\psi}$ is *minus* the precession rate about the $\mathbf{e}_{z'}$ axis, as seen in the body frame.

The full transformation between the fixed frame and the body frame is rather complicated. However, the following results can easily be verified:

$$\mathbf{e}_z = \sin\psi\,\sin\theta\,\mathbf{e}_{x'} + \cos\psi\,\sin\theta\,\mathbf{e}_{y'} + \cos\theta\,\mathbf{e}_{z'}, \tag{9.79}$$

$$\mathbf{e}_{x''} = \cos\psi\,\mathbf{e}_{x'} - \sin\psi\,\mathbf{e}_{y'}. \tag{9.80}$$

It follows from Eq. (9.79) that $\mathbf{e}_z\cdot\mathbf{e}_{z'} = \cos\theta$. In other words, θ is the angle of inclination between the z- and z'-axes. Finally, since the total angular velocity can be written

$$\boldsymbol{\omega} = \boldsymbol{\omega}_\phi + \boldsymbol{\omega}_\theta + \boldsymbol{\omega}_\psi, \tag{9.81}$$

Eqs. (9.74), (9.76), and (9.78)–(9.80) yield

$$\omega_{x'} = \sin\psi\,\sin\theta\,\dot{\phi} + \cos\psi\,\dot{\theta}, \tag{9.82}$$

$$\omega_{y'} = \cos\psi\,\sin\theta\,\dot{\phi} - \sin\psi\,\dot{\theta}, \tag{9.83}$$

$$\omega_{z'} = \cos\theta\,\dot{\phi} + \dot{\psi}. \tag{9.84}$$

The angles ϕ, θ, and ψ are termed *Eulerian angles*. Each has a clear physical interpretation: ϕ is the angle of precession about the $\mathbf{e}_z$ axis in the fixed frame, ψ is minus the angle of precession about the $\mathbf{e}_{z'}$ axis in the body frame, and θ is the angle of inclination between the $\mathbf{e}_z$ and $\mathbf{e}_{z'}$ axes. Moreover, we can express the components of the angular velocity vector $\boldsymbol{\omega}$ in the body frame entirely in terms of the Eulerian angles, and their time derivatives [see Eqs. (9.82)–(9.84)].

Consider a rigid body which is constrained to rotate about a fixed axis with the constant angular velocity ω. Let the fixed angular velocity vector point along the z-axis. In the previous subsection, we saw that the angular momentum and the torque were both steady in the body frame. Since there is no precession of quantities in the body frame, it follows that the Eulerian angle ψ is constant. Since the angular velocity vector is fixed in the body frame, as well as the fixed frame [as can be seen by applying Eq. (9.49) to $\boldsymbol{\omega}$ instead of $\mathbf{L}$], it must subtend a constant angle with the $\mathbf{e}_{z'}$ axis. Hence, the Eulerian angle θ is also constant. It follows from Eqs. (9.82)–(9.84) that

$$\omega_{x'} = \sin\psi\,\sin\theta\,\dot{\phi}, \tag{9.85}$$

$$\omega_{y'} = \cos\psi\,\sin\theta\,\dot{\phi}, \tag{9.86}$$

$$\omega_{z'} = \cos\theta\,\dot{\phi}, \tag{9.87}$$

which implies that $\omega \equiv (\omega_{x'}^2 + \omega_{y'}^2 + \omega_{z'}^2)^{1/2} = \dot{\phi}$. In other words, the precession rate, $\dot{\phi}$, in the fixed frame is equal to ω. Hence, in the fixed frame, the constant torque and angular momentum vectors found in the body frame *precess* about the

angular velocity vector (*i.e.*, about the z-axis) at the rate ω. As discussed in the previous subsection, for the special case where the angular velocity vector is *parallel* to one of the principal axes of the body, the angular momentum vector is parallel to the angular velocity vector, and the torque is *zero*. Thus, in this case, there is no precession in the fixed frame.

Consider a rotating device such as a flywheel or a propellor. If the device is *statically balanced* then its center of mass lies on the axis of rotation. This is desirable since, otherwise, gravity, which effectively acts at the center of mass, exerts a varying torque about the axis of rotation as the device rotates, giving rise to unsteady rotation. If the device is *dynamically balanced* then the axis of rotation is also a principal axis, so that, as the device rotates its angular momentum vector, $\mathbf{L}$, is parallel to the axis of rotation. This is desirable since, otherwise, the angular momentum vector is not parallel to the axis of rotation, and, therefore, precesses around it. Since $d\mathbf{L}/dt$ is equal to the torque, a precessing torque must also be applied to the device (at right-angles to both the axis and $\mathbf{L}$). The result is a reaction on the bearings which can give rise to violent vibration and wobbling, even when the device is statically balanced.

Consider a freely rotating body which is rotationally symmetric about one axis. In the absence of an external torque, the angular momentum vector $\mathbf{L}$ is a constant of the motion [see Eq. (9.3)]. Let $\mathbf{L}$ point along the z-axis. In the previous subsection, we saw that the angular momentum vector subtends a constant angle θ with the axis of symmetry: *i.e.*, with the z'-axis. Hence, the time derivative of the Eulerian angle θ is zero. We also saw that the angular momentum vector, the axis of symmetry, and the angular velocity vector are co-planar. Consider an instant in time at which all of these vectors lie in the y'-z' plane. This implies that $\omega_{x'} = 0$. According to the previous subsection, the angular momentum vector subtends a constant angle α with the symmetry axis. It follows that $\omega_{y'} = \omega \sin\alpha$ and $\omega_{z'} = \omega \cos\alpha$. Equation (9.82) yields $\psi = 0$. Hence, Eq. (9.83) yields

$$\omega \sin\alpha = \sin\theta\, \dot{\phi}. \tag{9.88}$$

This can be combined with Eq. (9.72) to give

$$\dot{\phi} = \omega \left[1 + \left(\frac{I_\parallel^2}{I_\perp^2} - 1\right) \cos^2\alpha\right]^{1/2}. \tag{9.89}$$

Finally, Eqs. (9.84), together with (9.72) and (9.88), yields

$$\dot{\psi} = \omega\,\cos\alpha - \cos\theta\,\dot{\phi} = \omega\,\cos\alpha\left(1 - \frac{\tan\alpha}{\tan\theta}\right) = \omega\,\cos\alpha\left(1 - \frac{I_\parallel}{I_\perp}\right). \tag{9.90}$$

A comparison of the above equation with Eq. (9.68) gives

$$\dot{\psi} = -\Omega. \tag{9.91}$$

Thus, as expected, $\dot{\psi}$ is minus the precession rate (of the angular momentum and angular velocity vectors) in the body frame. On the other hand, $\dot{\phi}$ is the precession rate (of the angular velocity vector and the symmetry axis) in the fixed frame. Note that $\dot{\phi}$ and Ω are quite dissimilar. For instance, Ω is negative for elongated bodies ($I_\parallel < I_\perp$) whereas $\dot{\phi}$ is positive definite. It follows that the precession is always in the same sense as L_z in the fixed frame, whereas the precession in the body frame is in the opposite sense to $L_{z'}$ for elongated bodies. We found, in the previous subsection, that for a *flattened* body the angular momentum vector lies between the angular velocity vector and the symmetry axis. This means that, in the fixed frame, the angular velocity vector and the symmetry axis lie on *opposite* sides of the fixed angular momentum vector. On the other hand, for an *elongated* body we found that the angular velocity vector lies between the angular momentum vector and the symmetry axis. This means that, in the fixed frame, the angular velocity vector and the symmetry axis lie on the *same* side of the fixed angular momentum vector.

As an example, consider the free rotation of a thin disk. It is easily demonstrated (from the perpendicular axis theorem) that

$$I_\parallel = 2\,I_\perp \tag{9.92}$$

for such a disk. Hence, from Eq. (9.68), the precession rate in the body frame is

$$\Omega = \omega\,\cos\alpha. \tag{9.93}$$

According to Eq. (9.89), the precession rate in the fixed frame is

$$\dot{\phi} = \omega\left[1 + 3\,\cos^2\alpha\right]^{1/2}. \tag{9.94}$$

In the limit in which α is small (*i.e.*, in which the angular velocity vector is almost parallel to the symmetry axis), we obtain

$$\Omega \simeq \omega, \tag{9.95}$$

$$\dot{\phi} \simeq 2\,\omega. \tag{9.96}$$

Thus, the symmetry axis precesses in the fixed frame at approximately twice the angular speed of rotation. This precession is manifest as a wobbling motion.

It is known that the axis of rotation of the Earth is very slightly inclined to its symmetry axis (which passes through the two poles). The angle α is approximately 0.2 seconds of an arc. It is also known that the ratio of the moments of inertia is about $I_\parallel/I_\perp = 1.00327$, as determined from the Earth's oblateness. Hence, from (9.68), the precession rate of the angular velocity vector about the symmetry axis, as viewed on Earth, is

$$\Omega = 0.00327\,\omega, \tag{9.97}$$

giving a precession period of

$$T' = \frac{2\pi}{\Omega} = 305 \text{ days}. \tag{9.98}$$

(Of course, $2\pi/\omega = 1$ day.) The observed period of precession is about 440 days. The disagreement between theory and observation is attributed to the fact that the Earth is not perfectly rigid. The (theroretical) precession rate of the Earth's symmetry axis, as viewed from space, is given by Eq. (9.89):

$$\dot{\phi} = 1.00327\,\omega. \tag{9.99}$$

The associated precession period is

$$T = \frac{2\pi}{\dot{\phi}} = 0.997 \text{ days}. \tag{9.100}$$

The free precession of the Earth's symmetry axis in space is superimposed on a much slower precession, with a period of about 26,000 years, due to the small gravitational torque exerted on the Earth by the Sun and the Moon, because of the Earth's slight oblateness.

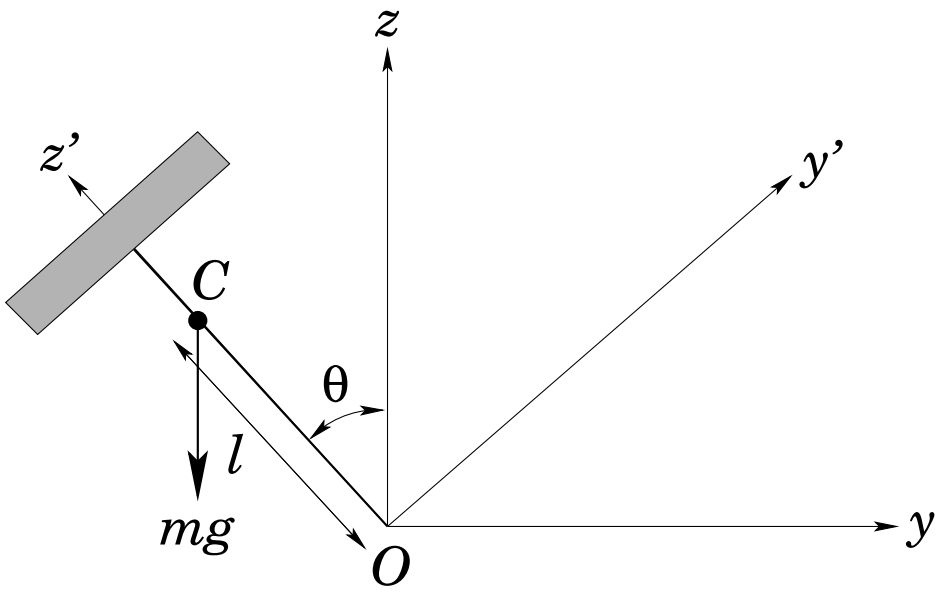

Figure 42:

9.9 Gyroscopic precession

Let us now study the motion of a *rotationally symmetric rigid top* which is free to turn about a fixed point (without friction), but which is subject to a gravitational torque (see Fig. 42). Let the z'-axis coincide with the symmetry axis. Let the principal moment of inertia about the symmetry axis be $I_\parallel$, and let the other principal moments both take the value $I_\perp$. Let the z-axis run vertically upwards, and let the common origin, O, of the fixed and body frames coincide with the fixed point about which the top turns. Suppose that the center of mass of the top lies a distance l along its axis for O, and that the mass of the top is m. Let the symmetry axis of the top subtend an angle θ (which is an Eulerian angle) with the upward vertical.

Consider an instant in time at which the Eulerian angle ψ is zero. This implies that the x'-axis is horizontal, as shown in the diagram. The gravitational force, which acts at the center of mass, thus exerts a torque $m\,g\,l\,\sin\theta$ in the x'-direction. Hence, the components of the torque in the body frame are

$$T_{x'} = m\,g\,l\,\sin\theta, \tag{9.101}$$

$$T_{y'} = 0, \tag{9.102}$$

$$T_{z'} = 0. \tag{9.103}$$

The components of the angular velocity vector in the body frame are given by Eqs. (9.82)–(9.84). Thus, Euler's equations (9.51)–(9.53) take the form:

$$m\,g\,l\,\sin\theta = I_\perp\,(\ddot{\theta} - \cos\theta\,\sin\theta\,\dot{\phi}^2) + L_\psi\,\sin\theta\,\dot{\phi}, \tag{9.104}$$

$$0 = I_\perp\,(2\,\cos\theta\,\dot{\theta}\,\dot{\phi} + \sin\theta\,\ddot{\phi}) - L_\psi\,\dot{\theta}, \tag{9.105}$$

$$0 = \dot{L_\psi}, \tag{9.106}$$

where

$$L_\psi = I_\parallel\,(\cos\theta\,\dot{\phi} + \dot{\psi}) = I_\parallel\,\omega_{z'}. \tag{9.107}$$

Multiplying Eq. (9.105) by $\sin\theta$, we obtain

$$\dot{L_\phi} = 0, \tag{9.108}$$

where

$$L_\phi = I_\perp\,\sin^2\theta\,\dot{\phi} + L_\psi\,\cos\theta. \tag{9.109}$$

According to Eqs. (9.106) and (9.108), the two quantities L_ψ and L_ϕ are constants of the motion. These two quantities are the *angular momenta* of the system about the z'- and z-axis, respectively. They are conserved because the gravitational torque has no component along either the z'- or the z-axis.

If there are no frictional forces acting on the top then the total energy, $E = K + U$, is also a constant of the motion. Now,

$$E = \frac{1}{2}\,\left(I_\perp\,\omega_{x'}^2 + I_\perp\,\omega_{y'}^2 + I_\parallel\,\omega_{z'}^2\right) + m\,g\,l\,\cos\theta. \tag{9.110}$$

When written in terms of the Eulerian angles (with $\psi = 0$), this becomes

$$E = \frac{1}{2}\,\left(I_\perp\,\dot{\theta}^2 + I_\perp\,\sin^2\theta\,\dot{\phi}^2 + L_\psi^2/I_\parallel\right) + m\,g\,l\,\cos\theta. \tag{9.111}$$

Eliminating $\dot{\phi}$ between Eqs. (9.109) and (9.111), we obtain the following differential equation for θ:

$$E = \frac{1}{2}\,I_\perp\,\dot{\theta}^2 + \frac{(L_\phi - L_\psi\,\cos\theta)^2}{2\,I_\perp\,\sin^2\theta} + \frac{1}{2}\frac{L_\psi^2}{I_\parallel} + m\,g\,l\,\cos\theta. \tag{9.112}$$

Let

$$E' = E - \frac{1}{2}\frac{L_\psi^2}{I_\parallel}, \tag{9.113}$$

and $u = \cos\theta$. It follows that

$$\dot{u}^2 = 2\,(E' - m\,g\,l\,u)\,(1-u^2)\,I_\perp^{-1} - (L_\phi - L_\psi\,u)^2\,I_\perp^{-2}, \tag{9.114}$$

or

$$\dot{u}^2 = f(u), \tag{9.115}$$

where $f(u)$ is a *cubic* polynomial. In principal, the above equation can be integrated to give u (and, hence, θ) as a function of t:

$$t = \int \frac{du}{\sqrt{f(u)}}. \tag{9.116}$$

Fortunately, we do not have to perform the above integration (which is very ugly) in order to discuss the general properties of the solution to Eq. (9.115). It is clear, from Eq. (9.116), that $f(u)$ needs to be *positive* in order to obtain a physical solution. Hence, the limits of the motion in θ are determined by the three roots of the equation $f(u) = 0$. Since θ must lie between 0 and $\pi/2$, it follows that u must lie between 0 and 1. It can easily be demonstrated that $f \rightarrow \pm\infty$ as $u \rightarrow \pm\infty$. It can also be shown that the largest root u_3 lies in the region $u_3 > 1$, and the two smaller roots u_1 and u_2 (if they exist) lie in the region $-1 \leq u \leq +1$. It follows that, in the region $-1 \leq u \leq 1$, $f(u)$ is only positive between u_1 and u_2. Figure 43 shows a case where u_1 and u_2 lie in the range 0 to 1. The corresponding values of θ—θ_1 and θ_2, say—are then the limits of the vertical motion. The axis of the top oscillates backward and forward between these two values of θ as the top precesses about the vertical axis. This oscillation is called *nutation*. Incidentally, if u_1 becomes negative then the nutation will cause the top to strike the ground (assuming that it is spinning on a level surface).

If there is a double root of $f(u) = 0$ (*i.e.*, if $u_1 = u_2$) then there is no nutation, and the top precesses steadily. However, the criterion for steady precession is most easily obtained directly from Eq. (9.104). In the absence of nutation, $\dot{\theta} = \ddot{\theta} = 0$. Hence, we obtain

$$m\,g\,l = -I_\perp\,\cos\theta\,\dot{\phi}^2 + L_\psi\,\dot{\phi}, \tag{9.117}$$

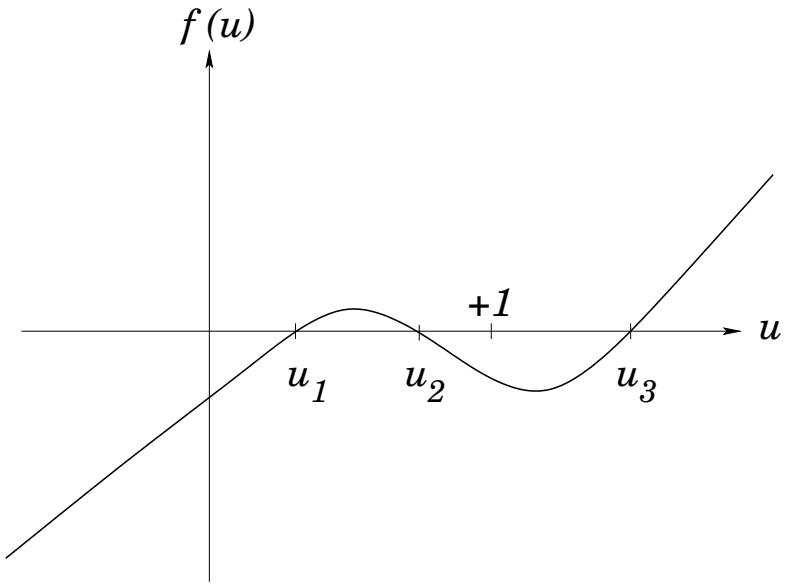

Figure 43:

or

$$L_\psi = \frac{m\,g\,l}{\dot\phi} + I_\perp\,\cos\theta\,\dot\phi. \tag{9.118}$$

The above equation is the criterion for steady precession. Since the right-hand side of Eq. (9.118) possesses the minimum value $2\,\sqrt{m\,g\,l\,I_\perp\,\cos\theta}$, it follows that

$$L_\psi > (L_\psi)_{\rm crit} = 2\,\sqrt{m\,g\,l\,I_\perp\,\cos\theta} \tag{9.119}$$

is a necessary condition for obtaining steady precession at the inclination angle θ. For $L_\psi > (L_\psi)_{\rm crit}$, there are two roots to Eq. (9.118), corresponding to a slow and a fast steady precession rate for a given inclination angle θ. If $L_\psi \gg (L_\psi)_{\rm crit}$ then these two roots are approximately given by

$$(\dot\phi)_{\rm slow} \simeq \frac{m\,g\,l}{L_\psi}, \tag{9.120}$$

$$(\dot\phi)_{\rm fast} \simeq \frac{L_\psi}{I_\perp\,\cos\theta}. \tag{9.121}$$

The slower of these two precession rates is the one which is generally observed.

9.10 Rotational stability

Consider a rigid body for which all of the principal moments of inertia are distinct. Let $I_{z'z'} > I_{y'y'} > I_{x'x'}$. Suppose that the body is freely rotating about one of its principal axes. What happens when the body is slightly disturbed?

Let the body be initially rotating about the x'-axis, so that

$$\boldsymbol{\omega} = \omega_{x'}\,\mathbf{e}_{x'}. \tag{9.122}$$

If we apply a slight perturbation then the angular velocity becomes

$$\boldsymbol{\omega} = \omega_{x'}\,\mathbf{e}_{x'} + \lambda\,\mathbf{e}_{y'} + \mu\,\mathbf{e}_{z'}, \tag{9.123}$$

where λ and μ are both assumed to be small. Euler's equations (9.51)–(9.53) take the form

$$I_{x'x'}\,\dot{\omega}_{x'} - (I_{y'y'} - I_{z'z'})\,\lambda\,\mu = 0, \tag{9.124}$$

$$I_{y'y'}\,\dot{\lambda} - (I_{z'z'} - I_{x'x'})\,\omega_{x'}\,\mu = 0, \tag{9.125}$$

$$I_{z'z'}\,\dot{\mu} - (I_{x'x'} - I_{y'y'})\,\omega_{x'}\,\lambda = 0. \tag{9.126}$$

Since $\lambda\,\mu$ is second-order in small quantities—and, therefore, negligible—the first of the above equations tells us that $\omega_{x'}$ is an approximate constant of the motion. The other two equations can be written

$$\dot{\lambda} = \left[\frac{(I_{z'z'} - I_{x'x'})\,\omega_{x'}}{I_{y'y'}}\right]\mu, \tag{9.127}$$

$$\dot{\mu} = -\left[\frac{(I_{y'y'} - I_{x'x'})\,\omega_{x'}}{I_{z'z'}}\right]\lambda. \tag{9.128}$$

Differentiating the first equation with respect to time, and then eliminating $\dot{\mu}$, we obtain

$$\ddot{\lambda} + \left[\frac{(I_{y'y'} - I_{x'x'})\,(I_{z'z'} - I_{x'x'})}{I_{y'y'}\,I_{z'z'}}\right]\omega_{x'}^2\,\lambda = 0. \tag{9.129}$$

It is easily demonstrated that μ satisfies the same differential equation. Since the term in square brackets in the above equation is *positive*, the equation takes the form of a *simple harmonic equation*, and, thus, has the bounded solution:

$$\lambda = \lambda_0\,\cos(\Omega_{x'}\,t - \alpha). \tag{9.130}$$

Here, λ_0 and α are constants of integration, and

$$\Omega_{x'} = \left[\frac{(I_{y'y'} - I_{x'x'})\,(I_{z'z'} - I_{x'x'})}{I_{y'y'}\,I_{z'z'}}\right]^{1/2} \omega_{x'}. \tag{9.131}$$

Thus, the body oscillates sinusoidally about its initial state with the angular frequency $\Omega_{x'}$. It follows that the body is *stable* to small perturbations when rotating about the x'-axis, in the sense that the amplitude of such perturbations does not grow in time.

Suppose that the body is initially rotating about the z'-axis, and is subject to a small perturbation. A similar argument to the above allows us to conclude that the body oscillates sinusoidally about its initial state with angular frequency

$$\Omega_{z'} = \left[\frac{(I_{z'z'} - I_{x'x'})\,(I_{z'z'} - I_{y'y'})}{I_{x'x'}\,I_{y'y'}}\right]^{1/2} \omega_{z'}. \tag{9.132}$$

Hence, the body is also stable to small perturbations when rotating about the z'-axis.

Suppose, finally, that the body is initially rotating about the y'-axis, and is subject to a small perturbation, such that

$$\boldsymbol{\omega} = \lambda\,\mathbf{e}_{x'} + \omega_{y'}\,\mathbf{e}_{y'} + \mu\,\mathbf{e}_{z'}. \tag{9.133}$$

It is easily demonstrated that λ satisfies the following differential equation:

$$\ddot{\lambda} - \left[\frac{(I_{y'y'} - I_{x'x'})\,(I_{z'z'} - I_{y'y'})}{I_{x'x'}\,I_{z'z'}}\right] \omega_{y'}^2\,\lambda = 0. \tag{9.134}$$

Note that the term in square brackets is *positive*. Hence, the above equation is *not* the simple harmonic equation. Indeed its solution takes the form

$$\lambda = A\,\mathrm{e}^{\,k\,t} + B\,\mathrm{e}^{-k\,t}. \tag{9.135}$$

Here, A and B are constants of integration, and

$$k = \left[\frac{(I_{y'y'} - I_{x'x'})\,(I_{z'z'} - I_{y'y'})}{I_{x'x'}\,I_{z'z'}}\right]^{1/2} \omega_{y'}. \tag{9.136}$$

In this case, the amplitude of the perturbation grows exponentially in time. Hence, the body is *unstable* to small perturbations when rotating about the y'-axis.

In conclusion, a rigid body with three distinct principal moments of inertia is stable to small perturbations when rotating about the principal axes with the largest and smallest moments, but is unstable when rotating about the axis with the intermediate moment.

Finally, if two of the principal moments are the same then it can be shown that the body is only stable to small perturbations when rotating about the principal axis whose moment is distinct from the other two.

10 Lagrangian dynamics

10.1 Introduction

In this section, we shall investigate an elegant reformulation of the laws of Newtonian dynamics due to the French/Italian mathematician Joseph Louis Lagrange. This reformulation is particularly useful for finding the equations of motion of complicated dynamical systems.

10.2 Generalized coordinates

Let the q_i, for $i = 1, \mathcal{F}$, be a set of coordinates which uniquely specifies the instantaneous configuration of some dynamical system. Here, it is assumed that each of the q_i can vary *independently*. The q_i might be Cartesian coordinates, or polar coordinates, or angles, or some mixture of all three types of coordinate, and are, therefore, termed *generalized coordinates*. A dynamical system whose instantaneous configuration is fully specified by $\mathcal{F}$ independent generalized coordinates is said to have $\mathcal{F}$ *degrees of freedom*. For instance, the instantaneous position of a particle moving freely in three dimensions is completely specified by its three Cartesian coordinates, x, y, and z. Moreover, these coordinates are clearly independent of one another. Hence, a dynamical system consisting of a single particle moving freely in three dimensions has three degrees of freedom. If there are two freely moving particles then the system has six degrees of freedom, and so on.

Suppose that we have a dynamical system consisting of N particles moving freely in three dimensions. This is an $\mathcal{F} = 3\,N$ degree of freedom system whose instantaneous configuration can be specified by $\mathcal{F}$ Cartesian coordinates. Let us denote these coordinates the x_j, for $j = 1, \mathcal{F}$. Thus, x_1, x_2, x_3 are the Cartesian coordinates of the first particle, x_4, x_5, x_6 the Cartesian coordinates of the second particle, *etc*. Suppose that the instantaneous configuration of the system can also be specified by $\mathcal{F}$ generalized coordinates, which we shall denote the q_i, for $i = 1, \mathcal{F}$. Thus, the q_i might be the polar coordinates of the particles. In general, we expect

the x_j to be functions of the q_i. In other words,

$$x_j = x_j(q_1, q_2, \cdots, q_{\mathcal{F}}, t) \tag{10.1}$$

for $j = 1, \mathcal{F}$. Here, for the sake of generality, we have included the possibility that the functional relationship between the x_j and the q_i might depend on the time t explicitly. This would be the case if the dynamical system were subject to time varying constraints. For instance, a system consisting of a particle constrained to move on a surface which is itself moving. Finally, by the chain rule, the variation of the x_j due to a variation of the q_i (at constant t) is given by

$$\delta x_j = \sum_{i=1,\mathcal{F}} \frac{\partial x_j}{\partial q_i}\, \delta q_i, \tag{10.2}$$

for $j = 1, \mathcal{F}$.

10.3 Generalized forces

The work done on the dynamical system when its Cartesian coordinates change by δx_j is simply

$$\delta W = \sum_{j=1,\mathcal{F}} f_j\, \delta x_j \tag{10.3}$$

Here, the f_j are the Cartesian components of the forces acting on the various particles making up the system. Thus, f_1, f_2, f_3 are the components of the force acting on the first particle, f_4, f_5, f_6 the components of the force acting on the second particle, *etc.* Using Eq. (10.2), we can also write

$$\delta W = \sum_{j=1,\mathcal{F}} f_j \sum_{i=1,\mathcal{F}} \frac{\partial x_j}{\partial q_i}\, \delta q_i. \tag{10.4}$$

The above expression can be rearranged to give

$$\delta W = \sum_{i=1,\mathcal{F}} Q_i\, \delta q_i, \tag{10.5}$$

where

$$Q_i = \sum_{j=1,\mathcal{F}} f_j\, \frac{\partial x_j}{\partial q_i}. \tag{10.6}$$

Here, the Q_i are termed *generalized forces*. Note that a generalized force does not necessarily have the dimensions of force. However, the product $Q_i\,q_i$ must have the dimensions of work. Thus, if a particular q_i is a Cartesian coordinate then the associated Q_i is a force. Conversely, if a particular q_i is an angle then the associated Q_i is a torque.

Suppose that the dynamical system in question is *conservative*. It follows that

$$f_j = -\frac{\partial U}{\partial x_j}, \tag{10.7}$$

for $j = 1, \mathcal{F}$, where $U(x_1, x_2, \cdots, x_{\mathcal{F}}, t)$ is the system's potential energy. Hence, according to Eq. (10.6),

$$Q_i = -\sum_{j=1,\mathcal{F}} \frac{\partial U}{\partial x_j}\,\frac{\partial x_j}{\partial q_i} = -\frac{\partial U}{\partial q_i}, \tag{10.8}$$

for $i = 1, \mathcal{F}$.

10.4 Lagrange's equation

The Cartesian equations of motion of our system take the form

$$m_j\,\ddot{x}_j = f_j, \tag{10.9}$$

for $j = 1, \mathcal{F}$, where m_1, m_2, m_3 are each equal to the mass of the first particle, m_4, m_5, m_6 are each equal to the mass of the second particle, *etc.* Furthermore, the kinetic energy of the system can be written

$$K = \frac{1}{2}\sum_{j=1,\mathcal{F}} m_j\,\dot{x}_j^2. \tag{10.10}$$

Now, since $x_j = x_j(q_1, q_2, \cdots, q_{\mathcal{F}}, t)$, we can write

$$\dot{x}_j = \sum_{i=1,\mathcal{F}} \frac{\partial x_j}{\partial q_i}\,\dot{q}_i + \frac{\partial x_j}{\partial t}, \tag{10.11}$$

for $j = 1, \mathcal{F}$. Hence, it follows that $\dot{x}_j = \dot{x}_j(\dot{q}_1, \dot{q}_2, \cdots, \dot{q}_\mathcal{F}, q_1, q_2, \cdots, q_\mathcal{F}, t)$. According to the above equation,

$$\frac{\partial \dot{x}_j}{\partial \dot{q}_i} = \frac{\partial x_j}{\partial q_i}, \tag{10.12}$$

where we are treating the $\dot{q}_i$ and the q_i as *independent* variables.

Multiplying Eq. (10.12) by $\dot{x}_j$, and then differentiating with respect to time, we obtain

$$\frac{d}{dt}\left(\dot{x}_j \frac{\partial \dot{x}_j}{\partial \dot{q}_i}\right) = \frac{d}{dt}\left(\dot{x}_j \frac{\partial x_j}{\partial q_i}\right) = \ddot{x}_j \frac{\partial x_j}{\partial q_i} + \dot{x}_j \frac{d}{dt}\left(\frac{\partial x_j}{\partial q_i}\right). \tag{10.13}$$

Now,

$$\frac{d}{dt}\left(\frac{\partial x_j}{\partial q_i}\right) = \sum_{k=1,\mathcal{F}} \frac{\partial^2 x_j}{\partial q_i \, \partial q_k} \, \dot{q}_k + \frac{\partial^2 x_j}{\partial q_i \, \partial t}. \tag{10.14}$$

Furthermore,

$$\frac{1}{2} \frac{\partial \dot{x}_j^2}{\partial \dot{q}_i} = \dot{x}_j \frac{\partial \dot{x}_j}{\partial \dot{q}_i}, \tag{10.15}$$

and

$$\begin{aligned} \frac{1}{2} \frac{\partial \dot{x}_j^2}{\partial q_i} = \dot{x}_j \frac{\partial \dot{x}_j}{\partial q_i} &= \dot{x}_j \frac{\partial}{\partial q_i}\left(\sum_{k=1,\mathcal{F}} \frac{\partial x_j}{\partial q_k} \, \dot{q}_k + \frac{\partial x_j}{\partial t}\right) \\ &= \dot{x}_j \left(\sum_{k=1,\mathcal{F}} \frac{\partial^2 x_j}{\partial q_i \, \partial q_k} \, \dot{q}_k + \frac{\partial^2 x_j}{\partial q_i \, \partial t}\right). \\ &= \dot{x}_j \frac{d}{dt}\left(\frac{\partial x_j}{\partial q_i}\right), \end{aligned} \tag{10.16}$$

where use has been made of Eq. (10.14). Thus, it follows from Eqs. (10.13), (10.15), and (10.16) that

$$\frac{d}{dt}\left(\frac{1}{2} \frac{\partial \dot{x}_j^2}{\partial \dot{q}_i}\right) = \ddot{x}_j \frac{\partial x_j}{\partial q_i} + \frac{1}{2} \frac{\partial \dot{x}_j^2}{\partial q_i}. \tag{10.17}$$

Let us take the above equation, multiply by m_j, and then sum over all j. We obtain

$$\frac{d}{dt}\left(\frac{\partial K}{\partial \dot{q}_i}\right) = \sum_{j=1,\mathcal{F}} f_j \frac{\partial x_j}{\partial q_i} + \frac{\partial K}{\partial q_i}, \tag{10.18}$$

where use has been made of Eqs. (10.9) and (10.10). Thus, it follows from Eq. (10.6) that

$$\frac{d}{dt}\left(\frac{\partial K}{\partial \dot{q}_i}\right) = Q_i + \frac{\partial K}{\partial q_i}. \tag{10.19}$$

Finally, making use of Eq. (10.8), we get

$$\frac{d}{dt}\left(\frac{\partial K}{\partial \dot{q}_i}\right) = -\frac{\partial U}{\partial q_i} + \frac{\partial K}{\partial q_i}. \tag{10.20}$$

It is helpful to introduce a function L, called the *Lagrangian*, which is defined as the difference between the kinetic and potential energies of the dynamical system under investigation:

$$L = K - U. \tag{10.21}$$

Since the potential energy U is clearly independent of the $\dot{q}_i$, it follows from Eq. (10.20) that

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0, \tag{10.22}$$

for $i = 1, \mathcal{F}$. This equation is known as *Lagrange's equation*.

According to the above analysis, if we can express the kinetic and potential energies of our dynamical system solely in terms of our generalized coordinates and their time derivatives, then we can immediately write down the equations of motion of the system, expressed in terms of the generalized coordinates, using Lagrange's equation, (10.22). Unfortunately, this scheme only works for conservative systems. Let us now consider some examples.

10.5 Motion in a central potential

Consider a particle of mass m moving in two dimensions in the central potential $U(r)$. This is clearly a two degree of freedom dynamical system. As described in Sect. 6.5, the particle's instantaneous position is most conveniently specified in terms of the plane polar coordinates r and θ. These are our two generalized coordinates. According to Eq. (6.14), the square of the particle's velocity can be written

$$v^2 = \dot{r}^{\,2} + (r\,\dot{\theta})^2. \tag{10.23}$$

Hence, the Lagrangian of the system takes the form

$$L = \frac{1}{2}\,m\,(\dot{r}^{\,2} + r^2\,\dot{\theta}^{\,2}) - U(r). \tag{10.24}$$

Note that

$$\frac{\partial L}{\partial \dot{r}} = m\,\dot{r}, \qquad \frac{\partial L}{\partial r} = m\,r\,\dot{\theta}^{\,2} - dU/dr, \tag{10.25}$$

$$\frac{\partial L}{\partial \dot{\theta}} = m\,r^2\,\dot{\theta}, \qquad \frac{\partial L}{\partial \theta} = 0. \tag{10.26}$$

Now, Lagrange's equation (10.22) yields the equations of motion,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) - \frac{\partial L}{\partial r} = 0, \tag{10.27}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0. \tag{10.28}$$

Hence, we obtain

$$\frac{d}{dt}(m\,\dot{r}) - m\,r\,\dot{\theta}^{\,2} + \frac{dU}{dr} = 0, \tag{10.29}$$

$$\frac{d}{dt}\big(m\,r^2\,\dot{\theta}\big) = 0, \tag{10.30}$$

or

$$\ddot{r} - r\,\dot{\theta}^{\,2} = -\frac{dV}{dr}, \tag{10.31}$$

$$r^2\,\dot{\theta} = h, \tag{10.32}$$

where $V = U/m$, and h is a constant. We can recognize Eqs. (10.31) and (10.32) as the equations we derived in Sect. 6 for motion in a central potential. The advantage of the Lagrangian method of deriving these equations is that we avoid having to express the acceleration in terms of the generalized coordinates r and θ.

10.6 Atwood machines

An Atwood machine consists of two weights, of mass m_1 and m_2, connected by a light inextensible cord of length l, which passes over a pulley of radius $a \ll l$, and moment of inertia I. See Fig. 44.

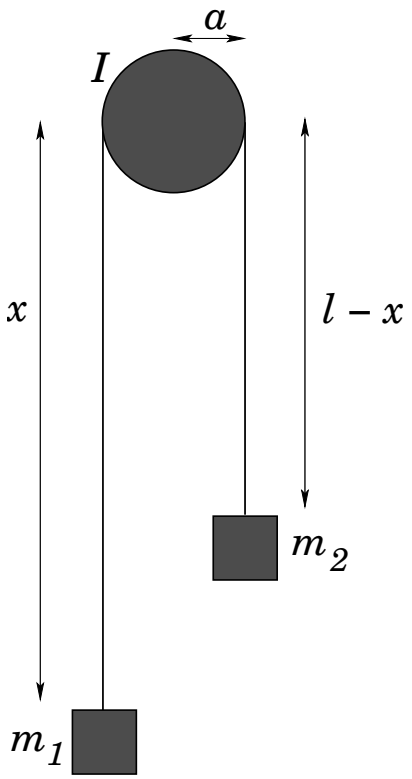

Figure 44:

Referring to the diagram, we can see that this is a one degree of freedom system whose instantaneous configuration is specified by the coordinate x. Assuming that the cord does not slip with respect to the pulley, the angular velocity of pulley is $\dot{x}/a$. Hence, the kinetic energy of the system is given by

$$K = \frac{1}{2}\, m_1\, \dot{x}^{\,2} + \frac{1}{2}\, m_2\, \dot{x}^{\,2} + \frac{1}{2}\, I\, \frac{\dot{x}^{\,2}}{a^2}. \tag{10.33}$$

The potential energy of the system takes the form

$$U = -m_1\, g\, x - m_2\, g\, (l - x). \tag{10.34}$$

It follows that the Lagrangian is written

$$\frac{1}{2}\left(m_1 + m_2 + \frac{I}{a^2}\right) \dot{x}^{\,2} + g\, (m_1 - m_2)\, x + \text{const.} \tag{10.35}$$

The equation of motion,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0, \tag{10.36}$$

thus yields

$$\left(m_1 + m_2 + \frac{I}{a^2}\right) \ddot{x} - g\, (m_1 - m_2) = 0, \tag{10.37}$$

or

$$\ddot{x} = \frac{g\, (m_1 - m_2)}{m_1 + m_2 + I/a^2}, \tag{10.38}$$

which is the correct answer.

Consider the dynamical system drawn in Fig. 45. This is an Attwood machine in which one of the weights has been replaced by a second Attwood machine. The system now has two degrees of freedom, and its instantaneous position is specified by the two coordinates x and x', as shown.

For the sake of simplicity, let us neglect the masses of the two pulleys. Thus, the kinetic energy of the system is written

$$K = \frac{1}{2}\, m_1\, \dot{x}^{\,2} + \frac{1}{2}\, m_2\, (-\dot{x} + \dot{x}')^2 + \frac{1}{2}\, m_3\, (-\dot{x} - \dot{x}')^2, \tag{10.39}$$

whereas the potential energy takes the form

$$U = -m_1\, g\, x - m_2\, g\, (l - x + x') - m_3\, g\, (l - x + l' - x'). \tag{10.40}$$

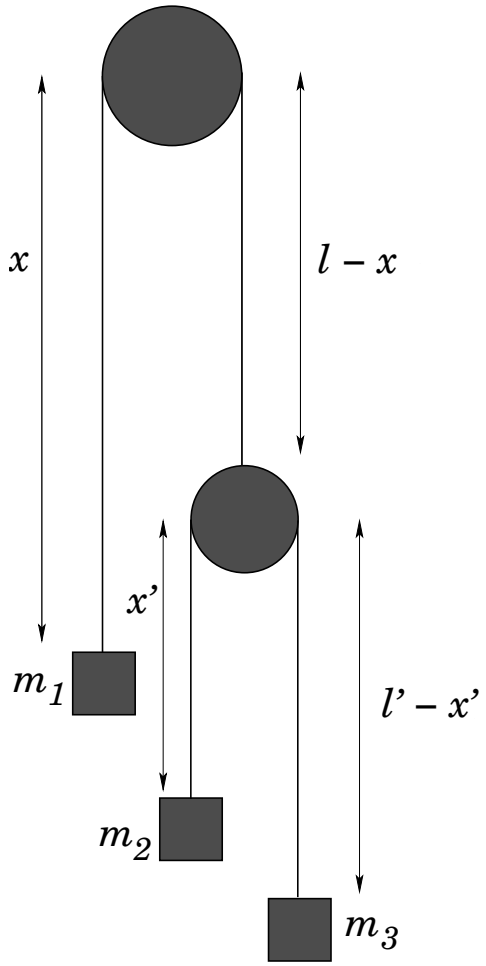

Figure 45:

It follows that the Lagrangian of the system is

$$\begin{aligned} L &= \frac{1}{2}\,m_1\,\dot{x}^{\,2} + \frac{1}{2}\,m_2\,(-\dot{x}+\dot{x}')^2 + \frac{1}{2}\,m_3\,(-\dot{x}-\dot{x}')^2 \\ &\quad + g\,(m_1 - m_2 - m_3)\,x + g\,(m_2 - m_3)\,x' + \text{constant}. \end{aligned} \tag{10.41}$$

Hence, the equations of motion,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0, \tag{10.42}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}'}\right) - \frac{\partial L}{\partial x'} = 0, \tag{10.43}$$

yield

$$m_1\,\ddot{x} + m_2\,(\ddot{x}-\ddot{x}') + m_3\,(\ddot{x}+\ddot{x}') - g\,(m_1 - m_2 - m_3) = 0, \tag{10.44}$$

$$m_2\,(-\ddot{x}+\ddot{x}') + m_3\,(\ddot{x}+\ddot{x}') - g\,(m_2 - m_3) = 0. \tag{10.45}$$

The accelerations $\ddot{x}$ and $\ddot{x}'$ can be obtained from the above two equations via simple algebra.

10.7 Sliding down a sliding plane

Consider the case of a particle of mass m sliding down a smooth inclined plane of mass M which is, itself, free to slide on a smooth horizontal surface, as shown in Fig. 46. This is a two degree of freedom system, so we need two coordinates to specify the configuration. Let us choose x, the horizontal distance of the plane from some reference point, and x', the parallel displacement of the particle from some reference point on the plane.

Defining x- and y-axes, as shown in the diagram, the x- and y-components of the particle's velocity are clearly given by

$$v_x = \dot{x} + \dot{x}'\,\cos\theta, \tag{10.46}$$

$$v_y = -\dot{x}'\,\sin\theta, \tag{10.47}$$

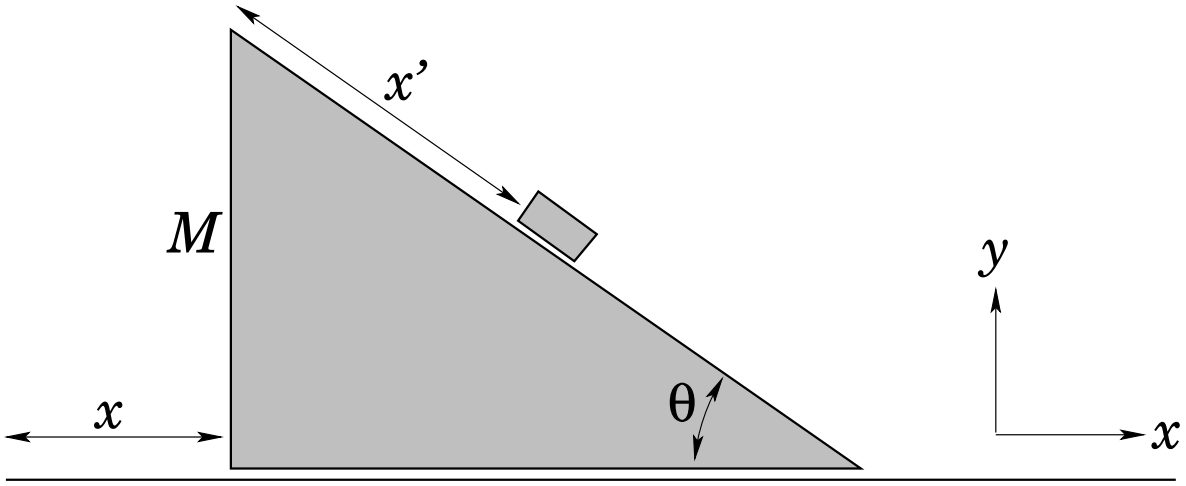

Figure 46:

respectively, where θ is the angle of inclination of the plane with respect to the horizontal. Thus,

$$v^2 = v_x^2 + v_y^2 = \dot{x}^{\,2} + 2\,\dot{x}\,\dot{x}'\,\cos\theta + \dot{x}'^{\,2}. \tag{10.48}$$

Hence, the kinetic energy of the system takes the form

$$K = \frac{1}{2}\,M\,\dot{x}^{\,2} + \frac{1}{2}\,m\,(\dot{x}^{\,2} + 2\,\dot{x}\,\dot{x}'\,\cos\theta + \dot{x}'^{\,2}), \tag{10.49}$$

whereas the potential energy is given by

$$U = -m\,g\,x'\,\sin\theta + \text{constant}. \tag{10.50}$$

It follows that the Lagrangian is written

$$L = \frac{1}{2}\,M\,\dot{x}^{\,2} + \frac{1}{2}\,m\,(\dot{x}^{\,2} + 2\,\dot{x}\,\dot{x}'\,\cos\theta + \dot{x}'^{\,2}) + m\,g\,x'\,\sin\theta + \text{const}. \tag{10.51}$$

The equations of motion,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0, \tag{10.52}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}'}\right) - \frac{\partial L}{\partial x'} = 0. \tag{10.53}$$

thus yield

$$M\,\ddot{x} + m\,(\ddot{x} + \ddot{x}'\,\cos\theta) = 0, \tag{10.54}$$

$$m\,(\ddot{x}' + \ddot{x}\,\cos\theta) - m\,g\,\sin\theta = 0. \tag{10.55}$$

Finally, solving for $\ddot{x}$ and $\ddot{x}'$, we obtain

$$\ddot{x} = -\frac{g \sin\theta \cos\theta}{(m+M)/m - \cos^2\theta}, \tag{10.56}$$

$$\ddot{x}' = \frac{g \sin\theta}{1 - m \cos^2\theta/(m+M)}. \tag{10.57}$$

10.8 Generalized momenta

Consider the motion of a single particle moving in one dimension. The kinetic energy is

$$K = \frac{1}{2} m \dot{x}^2, \tag{10.58}$$

where m is the mass of the particle, and x its displacement. Now, the particle's linear momentum is $p = m\dot{x}$. However, this can also be written

$$p = \frac{\partial K}{\partial \dot{x}} = \frac{\partial L}{\partial \dot{x}}, \tag{10.59}$$

since $L = K - U$, and the potential energy U is independent of $\dot{x}$.

Consider a dynamical system described by $\mathcal{F}$ generalized coordinates q_i, for $i = 1, \mathcal{F}$. By analogy with the above expression, we can define *generalized momenta* of the form

$$p_i = \frac{\partial L}{\partial \dot{q}_i}, \tag{10.60}$$

for $i = 1, \mathcal{F}$. Here, p_i is sometimes called the momentum *conjugate* to the coordinate q_i. Hence, Lagrange's equation (10.22) can be written

$$\frac{dp_i}{dt} = \frac{\partial L}{\partial q_i}, \tag{10.61}$$

for $i = 1, \mathcal{F}$. Note that a generalized momentum does not necessarily have the dimensions of linear momentum.

Suppose that the Lagrangian L does not depend explicitly on some coordinate q_k. It follows from Eq. (10.61) that

$$\frac{dp_k}{dt} = \frac{\partial L}{\partial q_k} = 0. \tag{10.62}$$

Hence,

$$p_k = \text{const.} \tag{10.63}$$

The coordinate q_k is said to be *ignorable* in this case. Thus, we conclude that the generalized momentum associated with an ignorable coordinate is a constant of the motion.

For example, in Sect. 10.5, the Lagrangian (10.24) for a particle moving in a central potential is independent of the angular coordinate θ. Thus, θ is an ignorable coordinate, and

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = m\, r^2\, \dot{\theta} \tag{10.64}$$

is a constant of the motion. Of course, p_θ is the angular momentum about the origin. This is conserved because a central force exerts no torque about the origin.

Again, in Sect. 10.7, the Lagrangian (10.51) for a mass sliding down a sliding slope is independent of the Cartesian coordinate x. It follows that x is an ignorable coordinate, and

$$p_x = \frac{\partial L}{\partial \dot{x}} = M\,\dot{x} + m\,(\dot{x} + \dot{x}'\,\cos\theta) \tag{10.65}$$

is a constant of the motion. Of course, p_x is the total linear momentum in the x-direction. This is conserved because there is no external force acting on the system in the x-direction.

10.9 The spherical pendulum

Consider a pendulum consisting of a mass m on the end of light inextensible string of length l. Suppose that the mass is free to move in any direction (as long as the string remains taut). Let the fixed end of the string be located at the origin of our coordinate system. We can define Cartesian coordinates, (x, y, z), such that the z-axis points vertically upward. We can also define spherical polar coordinates, (r, θ, ϕ), whose axis points along the $-z$-axis. The latter coordinates are the most convenient, since r is constrained to always take the value l. However, the two angular coordinates, θ and ϕ, are free to vary independently. Hence, this is clearly a two degree of freedom system.

The Cartesian coordinates can be written in terms of the angular coordinates θ and ϕ. In fact,

$$x = l \sin\theta \sin\phi, \tag{10.66}$$

$$y = l \sin\theta \cos\phi. \tag{10.67}$$

$$z = -l \cos\theta. \tag{10.68}$$

Hence, the potential energy of the system is

$$U = m\, g\, z = -m\, g\, l \cos\theta. \tag{10.69}$$

Also,

$$v^2 = \dot{x}^2 + \dot{y}^2 + \dot{z}^2 = l^2\,(\dot{\theta}^2 + \sin^2\theta\,\dot{\phi}^2). \tag{10.70}$$

Thus, the Lagrangian of the system is written

$$L = \frac{1}{2}\, m\, l^2\,(\dot{\theta}^2 + \sin^2\theta\,\dot{\phi}^2) + m\, g\, l \cos\theta. \tag{10.71}$$

Note that the Lagrangian is independent of the angular coordinate ϕ. It follows that

$$p_\phi = \frac{\partial L}{\partial \dot{\phi}} = m\, l^2 \sin^2\theta\,\dot{\phi} \tag{10.72}$$

is a constant of the motion. Of course, p_ϕ is the angular momentum of the system about the z-axis. This is conserved because neither the tension in the string nor the force of gravity exert a torque about the z-axis. Conservation of angular momentum about the z-axis implies that

$$\sin^2\theta\,\dot{\phi} = h, \tag{10.73}$$

where h is a constant.

The equation of motion of the system,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0, \tag{10.74}$$

yields

$$\ddot{\theta} + \frac{g}{l} \sin\theta - \sin\theta \cos\theta\,\dot{\phi}^2 = 0, \tag{10.75}$$

or

$$\ddot{\theta} + \frac{g}{l}\,\sin\theta - h^2\,\frac{\cos\theta}{\sin^3\theta} = 0, \tag{10.76}$$

where use has been made of Eq. (10.73).

Suppose that $\phi = \phi_0 = $ const. It follows that $\dot{\phi} = h = 0$. Hence, Eq. (10.76) yields

$$\ddot{\theta} + \frac{g}{l}\,\sin\theta = 0. \tag{10.77}$$

This, of course, is the equation of a simple pendulum whose motion is restricted to the vertical plane $\phi = \phi_0$ (see Sect. 4.9).

Suppose that $\theta = \theta_0 = $ const. It follows from Eq. (10.73) that $\dot{\phi} = \dot{\phi}_0 = $ const.: *i.e.*, the pendulum bob rotates uniformly in a horizontal plane. According to Eqs. (10.73) and (10.76),

$$\dot{\phi}_0 = \sqrt{\frac{g}{d}}, \tag{10.78}$$

where $d = l\,\cos\theta_0$ is the vertical distance of the plane of rotation below the pivot point. This type of pendulum is usually called a *conical pendulum*, since the string attached to the pendulum bob sweeps out a cone as the bob rotates.

Suppose, finally, that the motion is almost conical: *i.e.*, the value of θ remains close to the value θ_0. Let

$$\theta = \theta_0 + \delta\theta. \tag{10.79}$$

Taylor expanding Eq. (10.76) to first order in $\delta\theta$, the zeroth order terms cancel out, and we are left with

$$\delta\ddot{\theta} + \dot{\phi}_0^2\,(1 + 3\,\cos^2\theta_0)\,\delta\theta \simeq 0. \tag{10.80}$$

Hence, solving the above equation, we obtain

$$\theta \simeq \theta_0 + \delta\theta_0\,\cos(\Omega\,t), \tag{10.81}$$

where

$$\Omega = \dot{\phi}_0\,\sqrt{1 + 3\,\cos^2\theta_0}. \tag{10.82}$$

Thus, the angle θ executes simple harmonic motion about its mean value θ_0 at the angular frequency Ω.

Now the azimuthal angle, ϕ, increases by

$$\Delta\phi \simeq \dot{\phi}_0 \, \frac{\pi}{\Omega} = \frac{\pi}{\sqrt{1 + 3 \, \cos^2\theta_0}} \tag{10.83}$$

as the angle of inclination to the vertical, θ, goes between successive maxima and minima. Suppose that θ_0 is small. In this case, $\Delta\phi$ is slightly greater than $\pi/2$. Now if $\Delta\phi$ were exactly $\pi/2$ then the pendulum bob would trace out the outline of a slightly *warped circle*: *i.e.*, something like the outline of a potato chip or a saddle. The fact that $\Delta\phi$ is slightly greater than $\pi/2$ means that this shape *precesses* about the z-axis in the *same* direction as the direction rotation of the bob. The precession rate increases as the angle of inclination θ_0 increases. Suppose, now, that θ_0 is slightly less than $\pi/2$. (Of course, θ_0 can never exceed $\pi/2$). In this case, $\Delta\phi$ is slighly less than π. Now if $\Delta\phi$ were exactly π then the pendulum bob would trace out the outline of a slightly *tilted circle*. The fact that $\Delta\phi$ is slightly less than π means that this shape *precesses* about the z-axis in the *opposite* direction to the direction of rotation of the bob. The precession rate increases as the angle of inclination θ_0 decreases below $\pi/2$.

11 Hamiltonian dynamics

11.1 Introduction

In this section, we shall investigate the application of variational principles to classical dynamics.

11.2 The calculus of variations

It is a well-known fact, first enunciated by Archimedes, that the shortest distance between two points in a plane is a straight-line. However, suppose that we wish to demonstrate this result from first principles. Let us consider the length, l, of various curves, $y(x)$, which run between two fixed points, A and B, in a plane, as illustrated in Fig. 47. Now, l takes the form

$$l = \int_A^B [dx^2 + dy^2]^{1/2} = \int_a^b [1 + y'^2(x)]^{1/2}\, dx, \tag{11.1}$$

where $y' \equiv dy/dx$. Note that l is a function of the function $y(x)$. In mathematics, a function of a function is termed a *functional*.

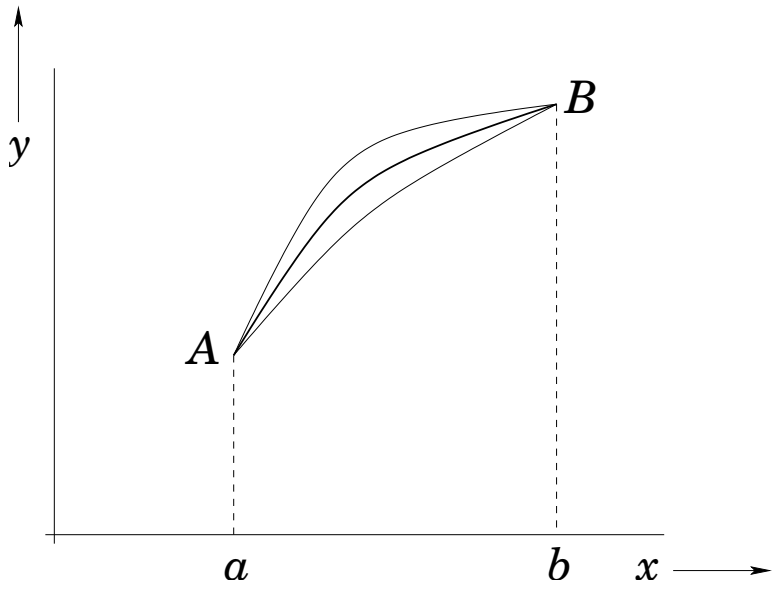

Figure 47:

Now, in order to find the shortest path between points A and B, we need to *minimize* the functional l with respect to small variations in the function $y(x)$, subject to the constraint that the end points, A and B, remain fixed. In other words, we need to solve

$$\delta l = 0. \tag{11.2}$$

The meaning of the above equation is that if $y(x) \rightarrow y(x) + \delta y(x)$, where $\delta y(x)$ is small, then the *first-order variation* in l, denoted δl, vanishes. In other words, $l \rightarrow l + O(\delta y^2)$. The particular function $y(x)$ for which $\delta l = 0$ obviously yields an *extremum* of l (*i.e.*, either a maximum or a minimum). Hopefully, in the case under consideration, it yields a minimum of l.

Consider a general functional of the form

$$I = \int_a^b F(y, y', x)\, dx, \tag{11.3}$$

where the end points of the integration are fixed. Suppose that $y(x) \rightarrow y(x)+\delta y(x)$. The first-order variation in I is written

$$\delta I = \int_a^b \left(\frac{\partial F}{\partial y}\, \delta y + \frac{\partial F}{\partial y'}\, \delta y' \right)\, dx, \tag{11.4}$$

where $\delta y' = d(\delta y)/dx$. Setting δI to zero, we obtain

$$\int_a^b \left(\frac{\partial F}{\partial y}\, \delta y + \frac{\partial F}{\partial y'}\, \delta y' \right)\, dx = 0. \tag{11.5}$$

This equation must be satisfied for all possible small perturbations $\delta y(x)$.

Integrating the second term in the integrand of the above equation by parts, we get

$$\int_a^b \left[\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'} \right) \right] \delta y\, dx + \left[\frac{\partial F}{\partial y'}\, \delta y \right]_a^b = 0. \tag{11.6}$$

Now, if the end points are fixed then $\delta y = 0$ at $x = a$ and $x = b$. Hence, the last term on the left-hand side of the above equation is zero. Thus, we obtain

$$\int_a^b \left[\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'} \right) \right] \delta y\, dx = 0. \tag{11.7}$$

The above equation must be satisfied for *all* small perturbations $\delta y(x)$. The only way in which this is possible is for the expression enclosed in square brackets in the integral to be zero. Hence, the functional I attains an extremum value whenever

$$\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) - \frac{\partial F}{\partial y} = 0. \tag{11.8}$$

This condition is known as the *Euler-Lagrange equation*.

Let us consider some special cases. Suppose that F does not explicitly depend on y. It follows that $\partial F/\partial y = 0$. Hence, the Euler-Lagrange equation (11.8) simplifies to

$$\frac{\partial F}{\partial y'} = \text{constant}. \tag{11.9}$$

Next, suppose that F does not depend explicitly on x. Multiplying Eq. (11.8) by y', we obtain

$$y'\,\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) - y'\,\frac{\partial F}{\partial y} = 0. \tag{11.10}$$

However,

$$\frac{d}{dx}\left(y'\,\frac{\partial F}{\partial y'}\right) = y'\,\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) + y''\,\frac{\partial F}{\partial y'}. \tag{11.11}$$

Thus, we get

$$\frac{d}{dx}\left(y'\,\frac{\partial F}{\partial y'}\right) = y'\,\frac{\partial F}{\partial y} + y''\,\frac{\partial F}{\partial y'}. \tag{11.12}$$

Now, if F is not an explicit function of x then the right-hand side of the above equation is the total derivative of F, namely dF/dx. Hence, we obtain

$$\frac{d}{dx}\left(y'\,\frac{\partial F}{\partial y'}\right) = \frac{dF}{dx}, \tag{11.13}$$

which yields

$$y'\,\frac{\partial F}{\partial y'} - F = \text{constant}. \tag{11.14}$$

Returning to the case under consideration, according to Eq. (11.1) and (11.3), we have $F = \sqrt{1+y'^2}$. Hence, F is not an explicit function of y, so Eq. (11.9) yields

$$\frac{\partial F}{\partial y'} = \frac{y'}{\sqrt{1+y'^2}} = c, \tag{11.15}$$

where c is a constant. So,

$$y' = \frac{c}{\sqrt{1-c^2}} = \text{constant}. \tag{11.16}$$

Of course, $y' =$ constant is the equation of a *straight-line*. Thus, the shortest distance between two fixed points in a plane is indeed a straight-line.

11.3 Conditional variation

Suppose that we wish to find the function $y(x)$ which maximizes or minimizes the functional

$$I = \int_a^b F(y, y', x)\, dx, \tag{11.17}$$

subject to the *constraint* that the value of

$$J = \int_a^b G(y, y', x)\, dx \tag{11.18}$$

remains constant. We can achieve our goal by finding an extremum of the new functional $K = I + \lambda\, J$, where $\lambda(x)$ is an undetermined function. We know that $\delta J = 0$, since the value of J is fixed, so if $\delta K = 0$ then $\delta I = 0$ as well. In other words, finding an extremum of K is equivalent to finding an extremum of I. Application of the Euler-Lagrange equation yields

$$\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) - \frac{\partial F}{\partial y} + \left[\frac{d}{dx}\left(\frac{\partial [\lambda\, G]}{\partial y'}\right) - \frac{\partial [\lambda\, G]}{\partial y}\right] = 0. \tag{11.19}$$

In principle, the above equation, together with the constraint (11.18), yields the functions $\lambda(x)$ and $y(x)$. Incidentally, λ is generally termed a *Lagrange multiplier*. If F and G have no explicit x-dependence then λ is usually a *constant*.

As an example, consider the following famous problem. Suppose that a uniform chain of fixed length l is suspended by its ends from two equal-height fixed points which are a distance a apart, where $a < l$. What is the equilibrium configuration of the chain?

Suppose that the chain has the uniform density per unit length ρ. Let the x- and y-axes be horizontal and vertical, respectively, and let the two ends of the chain lie

at $(\pm a/2, 0)$. The equilibrium configuration of the chain is specified by the function $y(x)$, for $-a/2 \leq x \leq +a/2$, where $y(x)$ is the vertical distance of the chain below its end points at horizontal position x. Of course, $y(-a/2) = y(+a/2) = 0$.

According to the discussion in Sect. 4.2, the stable equilibrium state of a conservative dynamical system is one which *minimizes* the system's potential energy. Now, the potential energy of the chain is written

$$U = -\rho\, g \int y\, ds = -\rho\, g \int_{-a/2}^{a/2} y\, [1 + y'^2]^{1/2}\, dx, \tag{11.20}$$

where $ds = \sqrt{dx^2 + dy^2}$ is an element of length along the chain, and g is the acceleration due to gravity. Hence, we need to minimize U with respect to small variations in $y(x)$. However, the variations in $y(x)$ must be such as to conserve the fixed length of the chain. Hence, our minimization procedure is subject to the constraint that

$$l = \int ds = \int_{-a/2}^{a/2} [1 + y'^2]^{1/2}\, dx \tag{11.21}$$

remains constant.

It follows, from the above discussion, that we need to minimize the functional

$$K = U + \lambda\, l = \int_{-a/2}^{a/2} (-\rho\, g\, y + \lambda)\, [1 + y'^2]^{1/2}\, dx, \tag{11.22}$$

where λ is an, as yet, undetermined constant. Since the integrand in the functional does not depend explicitly on x, we have from Eq. (11.14) that

$$y'^2\, (-\rho\, g\, y + \lambda)\, [1 + y'^2]^{-1/2} - (-\rho\, g\, y + \lambda)\, [1 + y'^2]^{1/2} = k, \tag{11.23}$$

where k is a constant. This expression reduces to

$$y'^2 = \left(\lambda' + \frac{y}{h}\right)^2 - 1, \tag{11.24}$$

where $\lambda' = \lambda/k$, and $h = -k/\rho\, g$.

Let

$$\lambda' + \frac{y}{h} = -\cosh z. \tag{11.25}$$

Making this substitution, Eq. (11.24) yields

$$\frac{dz}{dx} = -h^{-1}. \tag{11.26}$$

Hence,

$$z = -\frac{x}{h} + c, \tag{11.27}$$

where c is a constant. It follows from Eq. (11.25) that

$$y(x) = -h\,[\lambda' + \cosh(-x/h + c)]. \tag{11.28}$$

The above solution contains three undetermined constants, h, λ', and c. We can eliminate two of these constants by application of the boundary conditions $y(\pm a/2) = 0$. This yields

$$\lambda' + \cosh(\mp a/2\,h + c) = 0. \tag{11.29}$$

Hence, $c = 0$, and $\lambda' = -\cosh(a/2\,h)$. It follows that

$$y(x) = h\,[\cosh(a/2\,h) - \cosh(x/h)]. \tag{11.30}$$

The final unknown constant, h, is determined via the application of the constraint (11.21). Thus,

$$l = \int_{-a/2}^{a/2} [1 + y'^2]^{1/2}\,dx = \int_{-a/2}^{a/2} \cosh(x/h)\,dx = 2\,h\,\sinh(a/2\,h). \tag{11.31}$$

Hence, the equilibrium configuration of the chain is given by the curve (11.30), which is known as a *catenary*, where the parameter h satisfies

$$\frac{l}{2\,h} = \sinh\left(\frac{a}{2\,h}\right). \tag{11.32}$$

11.4 Multi-function variation

Suppose that we wish to maximize or minimize the functional

$$I = \int_a^b F(y_1, y_2, \cdots, y_N, y_1', y_2', \cdots, y_N', x)\,dx. \tag{11.33}$$

Here, the integrand F is now a functional of the N independent functions $y_i(x)$, for $i = 1, N$. A fairly straightforward extension of the analysis in Sect. 11.2 yields N separate Euler-Lagrange equations,

$$\frac{d}{dx}\left(\frac{\partial F}{\partial y_i'}\right) - \frac{\partial F}{\partial y_i} = 0, \tag{11.34}$$

for $i = 1, N$, which determine the N functions $y_i(x)$. If F does not explicitly depend on the function y_k then the kth Euler-Lagrange equation simplifies to

$$\frac{\partial F}{\partial y_k'} = \text{constant}. \tag{11.35}$$

Likewise, if F does not explicitly depend on x then all N Euler-Lagrange equations simplify to

$$y_i' \frac{\partial F}{\partial y_i'} - F = \text{constant}, \tag{11.36}$$

for $i = 1, N$.

11.5 Hamilton's principle

We saw, in Sect. 10, that we can specify the instantaneous configuration of a conservative dynamical system with $\mathcal{F}$ degrees of freedom in terms of $\mathcal{F}$ independent generalized coordinates q_i, for $i = 1, \mathcal{F}$. Let $K(q_1, q_2, \cdots, q_{\mathcal{F}}, \dot{q}_1, \dot{q}_2, \cdots, \dot{q}_{\mathcal{F}}, t)$ and $U(q_1, q_2, \cdots, q_{\mathcal{F}}, t)$ represent the kinetic and potential energies of the system, respectively, expressed in terms of these generalized coordinates. Here, $\dot{} \equiv d/dt$. The Lagrangian of the system is defined

$$L(q_1, q_2, \cdots, q_{\mathcal{F}}, \dot{q}_1, \dot{q}_2, \cdots, \dot{q}_{\mathcal{F}}, t) = K - U. \tag{11.37}$$

Finally, the $\mathcal{F}$ Lagrangian equations of motion of the system take the form

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0, \tag{11.38}$$

for $i = 1, \mathcal{F}$.

Note that the above equations of motion have exactly the same mathematical form as the Euler-Lagrange equations (11.34). Indeed, it is clear, from Sect. 11.4,

that the $\mathcal{F}$ Lagrangian equations of motion (11.38) can all be derived from a single equation: namely,

$$\delta \int_{t_1}^{t_2} L(q_1, q_2, \cdots, q_{\mathcal{F}}, \dot{q}_1, \dot{q}_2, \cdots, \dot{q}_{\mathcal{F}}, t)\, dt = 0. \tag{11.39}$$

In other words, the motion of the system in a given time interval is such as to maximize or minimize the time integral of the Lagrangian, which is know as the *action integral*. Thus, all of Newtonian mechanics can be summarized in a single statement:

> The motion of a dynamical system in a given time interval is such as to maximize or minimize the action integral.

(In practice, the action integral is almost always minimized.) This statement is known as *Hamilton's principle*, and was first formulated in 1834 by the Irish mathematician William Hamilton.

11.6 Constrained Lagrangian dynamics

Suppose that we have a dynamical system described by two generalized coordinates, q_1 and q_2. Suppose, further, that q_1 and q_2 are *not* independent variables. In other words, q_1 and q_2 are connected via some constraint equation of the form

$$f(q_1, q_2, t) = 0. \tag{11.40}$$

Let $L(q_1, q_2, \dot{q}_1, \dot{q}_2, t)$ be the Lagrangian. How do we write the Lagrangian equations of motion of the system?

Well, according to Hamilton's principle,

$$\delta \int_{t_1}^{t_2} L\, dt = \int_{t_1}^{t_2} \left\{ \left[\frac{\partial L}{\partial q_1} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_1} \right) \right] \delta q_1 + \left[\frac{\partial L}{\partial q_2} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_2} \right) \right] \delta q_2 \right\} dt = 0. \tag{11.41}$$

However, at any given instant in time, δq_1 and δq_2 are not independent. Indeed, Eq. (11.40) yields

$$\delta f = \frac{\partial f}{\partial q_1}\, \delta q_1 + \frac{\partial f}{\partial q_2}\, \delta q_2 = 0 \tag{11.42}$$

at a fixed time. Eliminating δq_2 from Eq. (11.41), we obtain

$$\int_{t_1}^{t_2} \left\{ \left[\frac{\partial L}{\partial q_1} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_1}\right)\right] \frac{1}{\partial f/\partial q_1} - \left[\frac{\partial L}{\partial q_2} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_2}\right)\right] \frac{1}{\partial f/\partial q_2} \right\} \delta q_1 \, dt = 0. \quad (11.43)$$

This equation must be satisfied for all possible perturbations $\delta q_1(t)$, which implies that the term enclosed in curly brackets is zero. Hence, we obtain

$$\frac{\partial L/\partial q_1 - (d/dt)\,(\partial L/\partial \dot{q}_1)}{\partial f/\partial q_1} = \frac{\partial L/\partial q_2 - (d/dt)\,(\partial L/\partial \dot{q}_2)}{\partial f/\partial q_2}. \quad (11.44)$$

One obvious way in which we can solve this equation is to separately set both sides equal to the same function of time, which we shall denote $-\lambda(t)$. It follows that the Lagrangian equations of motion of the system can be written

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_1}\right) - \frac{\partial L}{\partial q_1} - \lambda(t)\,\frac{\partial f}{\partial q_1} = 0, \quad (11.45)$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_2}\right) - \frac{\partial L}{\partial q_2} - \lambda(t)\,\frac{\partial f}{\partial q_2} = 0. \quad (11.46)$$

In principle, the above two equations can be solved, together with the constraint equation (11.40), to give $q_1(t)$, $q_2(t)$, and the Lagrange multiplier $\lambda(t)$. Now, the generalized force conjugate to the generalized coordinate q_1 is written [see Eqs. (10.8) and (10.21)]

$$Q_1 = \frac{\partial L}{\partial q_1}. \quad (11.47)$$

By analogy, it is clear from Eq. (11.45) that the generalized *constraint force* [*i.e.*, the generalized force responsible for maintaining the constraint (11.40)] conjugate to q_1 takes the form

$$\tilde{Q}_1 = \lambda(t)\,\frac{\partial f}{\partial q_1}, \quad (11.48)$$

with a similar expression for the generalized constraint force conjugate to q_2.

Suppose, now, that we have a dynamical system described by $\mathcal{F}$ generalized coordinates q_i, for $i = 1, \mathcal{F}$, which is subject to the constraint

$$f(q_1, q_2, \cdots, q_{\mathcal{F}}, t) = 0. \quad (11.49)$$

A simple extension of above analysis yields following the Lagrangian equations of motion of the system,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} - \lambda(t)\,\frac{\partial f}{\partial q_i} = 0, \tag{11.50}$$

for $i = 1, \mathcal{F}$. As before,

$$\tilde{Q}_i = \lambda(t)\,\frac{\partial f}{\partial q_i} \tag{11.51}$$

is the generalized constraint force conjugate to q_i.

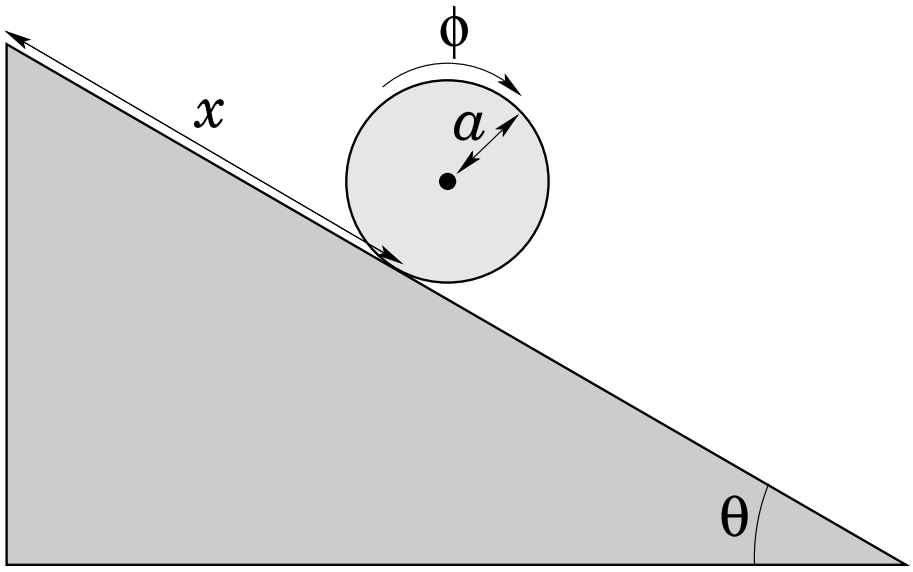

Figure 48:

Consider the following example. A cylinder of radius a rolls without slipping down a plane inclined at an angle θ to the horizontal. Let x represent the downward displacement of the center of mass of the cylinder parallel to the surface of the plane, and let ϕ represent the angle of rotation of the cylinder about its symmetry axis. The fact that the cylinder is rolling without slipping implies that x and ϕ are interrelated via the well-known constraint

$$f = x - a\,\theta = 0. \tag{11.52}$$

The Lagrangian of the cylinder takes the form

$$L = \frac{1}{2}\,m\,\dot{x}^{\,2} + \frac{1}{2}\,I\,\dot{\phi}^{\,2} + m\,g\,x\,\sin\theta, \tag{11.53}$$

where m is the cylinder's mass, I its moment of inertia, and g the acceleration due to gravity.

Note that $\partial f/\partial x = 1$ and $\partial f/\partial \phi = -a$. Hence, Eq. (11.50) yields the following Lagrangian equations of motion:

$$m\,\ddot{x} - m\,g\,\sin\theta - \lambda = 0, \tag{11.54}$$

$$I\,\ddot{\phi} + \lambda\,a = 0. \tag{11.55}$$

Equations (11.52), (11.54), and (11.55) can be solved to give

$$\ddot{x} = \frac{g\,\sin\theta}{1 + I/m\,a^2}, \tag{11.56}$$

$$a\,\ddot{\phi} = \frac{g\,\sin\theta}{1 + I/m\,a^2}, \tag{11.57}$$

$$\lambda = -\frac{m\,g\,\sin\theta}{1 + m\,a^2/I}. \tag{11.58}$$

The generalized constraint force conjugate to x is

$$\tilde{Q}_x = \lambda\,\frac{\partial f}{\partial x} = -\frac{m\,g\,\sin\theta}{1 + m\,a^2/I}. \tag{11.59}$$

This represents the frictional force acting parallel to the plane which impedes the downward acceleration of the cylinder, causing it to be less than the standard value $m\,g\,\sin\theta$. The generalized constraint force conjugate to ϕ is

$$\tilde{Q}_\phi = \lambda\,\frac{\partial f}{\partial \phi} = \frac{m\,g\,a\,\sin\theta}{1 + m\,a^2/I}. \tag{11.60}$$

This represents the frictional torque acting on the cylinder which forces the cylinder to rotate in such a manner that the constraint (11.52) is always satisfied.

Consider a second example. A bead of mass m slides without friction on a vertical circular hoop of radius a. Let r be the radial coordinate of the bead, and let θ be its angular coordinate, with the lowest point on the hoop corresponding to $\theta = 0$. Both coordinates are measured relative to the center of the hoop. Now, the bead is constrained to slide along the wire, which implies that

$$f = r - a = 0. \tag{11.61}$$

Note that $\partial f/\partial r = 1$ and $\partial f/\partial\theta = 0$. The Lagrangian of the system takes the form

$$L = \frac{1}{2}\,m\,(\dot{r}^{\,2} + r^2\,\dot{\theta}^{\,2}) + m\,g\,r\,\cos\theta. \tag{11.62}$$

Hence, according to Eq. (11.50), the Lagrangian equations of motion of the system are written

$$m\,\ddot{r} - m\,r\,\dot{\theta}^{\,2} - m\,g\,\cos\theta - \lambda = 0, \tag{11.63}$$

$$m\,r^2\,\ddot{\theta} + m\,g\,r\,\sin\theta = 0. \tag{11.64}$$

The second of these equations can be integrated (by multiplying by $\dot{\theta}$), subject to the constraint (11.61), to give

$$\dot{\theta}^{\,2} = \frac{2\,g}{a}\,\cos\theta + c, \tag{11.65}$$

where c is a constant. Let v_0 be the tangential velocity of the bead at the bottom of the hoop (*i.e.*, at $\theta = 0$). It follows that

$$\dot{\theta}^{\,2} = \frac{2\,g}{a}\,(\cos\theta - 1) + \frac{v_0^2}{a^2}. \tag{11.66}$$

Equations (11.61), (11.63), and (11.66) can be combined to give

$$\lambda = -m\left(3\,g\,\cos\theta - 2\,g + \frac{v_0^2}{a}\right). \tag{11.67}$$

Finally, the constraint force conjugate to r is given by

$$\tilde{Q}_r = \lambda\,\frac{\partial f}{\partial r} = -m\left(3\,g\,\cos\theta - 2\,g + \frac{v_0^2}{a}\right). \tag{11.68}$$

This represents the radial reaction exerted on the bead by the hoop. Of course, there is no constraint force conjugate to θ (since $\partial f/\partial\theta = 0$) because the bead slides without friction.

11.7 Hamilton's equations

Consider a dynamical system with $\mathcal{F}$ degrees of freedom described by the generalized coordinates q_i, for $i = 1, \mathcal{F}$. Suppose that neither the kinetic energy K nor

the potential energy U depend explicitly on the time t. Now, in conventional dynamical systems, the potential energy is generally independent of the $\dot{q}_i$, whereas the kinetic energy takes the form of a *homogeneous quadratic function* of the $\dot{q}_i$. In other words,

$$K = \sum_{i,j=1,\mathcal{F}} m_{ij}\,\dot{q}_i\,\dot{q}_j, \tag{11.69}$$

where the m_{ij} depend on the q_i, but not on the $\dot{q}_i$. It is easily demonstrated from the above equation that

$$\sum_{i=1,\mathcal{F}} \dot{q}_i \frac{\partial K}{\partial \dot{q}_i} = 2\,K. \tag{11.70}$$

Recall, from Sect. 10.8, that generalized momentum conjugate to the ith generalized coordinate is defined

$$p_i = \frac{\partial L}{\partial \dot{q}_i} = \frac{\partial K}{\partial \dot{q}_i}, \tag{11.71}$$

where $L = K - U$ is the Lagrangian of the system, and we have made use of the fact that U is independent of the $\dot{q}_i$. Consider the function

$$H = \sum_{i=1,\mathcal{F}} \dot{q}_i\,p_i - L = \sum_{i=1,\mathcal{F}} \dot{q}_i\,p_i - K + U. \tag{11.72}$$

If all of the conditions discussed above are satisfied, then Eqs. (11.70) and (11.71) yield

$$H = K + U. \tag{11.73}$$

In other words, the function H is equal to the *total energy* of the system.

Consider the variation of the function H. We have

$$\delta H = \sum_{i=1,\mathcal{F}} \left(\delta\dot{q}_i\,p_i + \dot{q}_i\,\delta p_i - \frac{\partial L}{\partial \dot{q}_i}\,\delta\dot{q}_i - \frac{\partial L}{\partial q_k}\,\delta q_k \right). \tag{11.74}$$

The first and third terms in the bracket cancel, because $p_i = \partial L/\partial\dot{q}_i$. Furthermore, since Lagrange's equation can be written $\dot{p}_i = \partial L/\partial q_i$ [see Sect. 10.8], we obtain

$$\delta H = \sum_{i=1,\mathcal{F}} (\dot{q}_i\,\delta p_i - \dot{p}_i\,\delta q_i). \tag{11.75}$$

Suppose, now, that we can express the total energy of the system, H, solely as a function of the q_i and the p_i, with no explicit dependence on the $\dot{q}_i$. In other

words, suppose that we can write $H = H(q_i, p_i)$. When the energy is written in this fashion it is generally termed the *Hamiltonian* of the system. The variation of the Hamiltonian function takes the form

$$\delta H = \sum_{i=1,\mathcal{F}} \left(\frac{\partial H}{\partial p_i}\, \delta p_i + \frac{\partial H}{\partial q_i}\, \delta q_i \right). \tag{11.76}$$

A comparison of the previous two equations yields

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \tag{11.77}$$

$$\dot{p}_i = -\frac{\partial H}{\partial q_i}, \tag{11.78}$$

for $i = 1, \mathcal{F}$. These $2\mathcal{F}$ first-order differential equations are known as *Hamilton's equations*. Hamilton's equations are often a useful alternative to Lagrange's equations, which take the form of $\mathcal{F}$ second-order differential equations.

Consider a one-dimensional harmonic oscillator. The kinetic and potential energies of the system are written $K = (1/2)\, m\, \dot{x}^{\,2}$ and $U = (1/2)\, k\, x^2$, where x is the displacement, m the mass, and $k > 0$. The generalized momentum conjugate to x is

$$p = \frac{\partial K}{\partial \dot{x}} = m\, \dot{x}. \tag{11.79}$$

Hence, we can write

$$K = \frac{1}{2}\, \frac{p^2}{m}. \tag{11.80}$$

So, the Hamiltonian of the system takes the form

$$H = K + U = \frac{1}{2}\, \frac{p^2}{m} + \frac{1}{2}\, k\, x^2. \tag{11.81}$$

Thus, Hamilton's equations, (11.77) and (11.78), yield

$$\dot{x} = \frac{\partial H}{\partial p} = \frac{p}{m}, \tag{11.82}$$

$$\dot{p} = -\frac{\partial H}{\partial x} = -k\, x. \tag{11.83}$$

Of course, the first equation is just a restatement of Eq. (11.79), whereas the second is Newton's second law of motion for the system.

Consider a particle of mass m moving in the central potential $U(r)$. In this case,

$$K = \frac{1}{2}\, m\, (\dot{r}^{\,2} + r^2\, \dot{\theta}^{\,2}), \tag{11.84}$$

where r, θ are plane polar coordinates. The generalized momenta conjugate to r and θ are

$$p_r = \frac{\partial K}{\partial \dot{r}} = m\, \dot{r}, \tag{11.85}$$

$$p_\theta = \frac{\partial K}{\partial \dot{\theta}} = m\, r^2\, \dot{\theta}, \tag{11.86}$$

respectively. Hence, we can write

$$K = \frac{1}{2\, m}\left(p_r^2 + \frac{p_\theta^2}{r^2}\right). \tag{11.87}$$

Thus, the Hamiltonian of the system takes the form

$$K = \frac{1}{2\, m}\left(p_r^2 + \frac{p_\theta^2}{r^2}\right) + U(r). \tag{11.88}$$

In this case, Hamilton's equations yield

$$\dot{r} = \frac{\partial H}{\partial p_r} = \frac{p_r}{m}, \tag{11.89}$$

$$\dot{\theta} = \frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{m\, r^2}, \tag{11.90}$$

which are just restatements of Eqs. (11.85) and (11.86), respectively, as well as

$$\dot{p}_r = -\frac{\partial H}{\partial r} = \frac{p_\theta^2}{m\, r^3} - \frac{\partial U}{\partial r}, \tag{11.91}$$

$$\dot{p}_\theta = -\frac{\partial H}{\partial \theta} = 0. \tag{11.92}$$

The last equation implies that

$$\frac{p_\theta}{m} = r^2\, \dot{\theta} = h, \tag{11.93}$$

where h is a constant. This can be combined with Eq. (11.91) to give

$$\frac{\dot{p}_r}{m} = \ddot{r} = \frac{h^2}{r^3} - \frac{\partial V}{\partial r}, \tag{11.94}$$

where $V = U/r$. Of course, Eqs. (11.93) and (11.94) are the conventional equations of motion for a particle moving in a central potential (see Sect. 6).

12 Coupled oscillations

12.1 Introduction

In this section, we shall investigate the dynamics of a many degree of freedom dynamical system which is perturbed from some equilibrium state.

12.2 Equilibrium state

Consider an $\mathcal{F}$ degree of freedom dynamical system described by the generalized coordinates q_i, for $i = 1, \mathcal{F}$. Suppose that the kinetic energy K and the potential energy U are not explicit functions of time. This implies that the system in question is *isolated*: *i.e.*, it is not subject to any external forces or time-varying constraints. In virtually all dynamical systems of interest, the kinetic energy can be expressed as a *quadratic form*: *i.e.*,

$$K = \frac{1}{2} \sum_{i,j=1,\mathcal{F}} m_{ij}(q_1, q_2, \cdots, q_{\mathcal{F}})\, \dot{q}_i\, \dot{q}_j. \tag{12.1}$$

Without loss of generality, we can specify that the weights m_{ij} in the above form are invariant under interchange of the indices i and j: *i.e.*,

$$m_{ij} = m_{ji}. \tag{12.2}$$

Finally, the potential energy is written $U = U(q_1, q_2, \cdots, q_{\mathcal{F}})$.

Suppose that $q_i = q_{i\,0}$, for $i = 1, \mathcal{F}$, corresponds to an *equilibrium state* of the system. It follows that $q_i = q_{i\,0}$ and $\dot{q}_i = \ddot{q}_i = 0$, for $i = 1, \mathcal{F}$, should be a possible solution of the equations of motion. Now, Lagrange's equations of motion for the system take the form [see Eq. (10.22)]

$$\frac{d}{dt}\left(\frac{\partial K}{\partial \dot{q}_i}\right) - \frac{\partial K}{\partial q_i} + \frac{\partial U}{\partial q_i} = 0, \tag{12.3}$$

for $i = 1, \mathcal{F}$. Here, we have made use of the definition $L = K - U$, and the fact that U is independent of the $\dot{q}_i$. Now, it is clear, from an examination of Eq. (12.1),

that every component making up the first two terms in the above equation contains either a $\dot{q}_j$ or a $\ddot{q}_j$, for some j. But, we can set all of the generalized velocities and accelerations to zero in an equilibrium state of the system. Hence, the first two terms in the above equation are zero, and the condition for equilibrium reduces to

$$Q_{i\,0} = -\frac{\partial U(q_{1\,0}, q_{2\,0}, \cdots, q_{\mathcal{F}\,0})}{\partial q_i} = 0, \tag{12.4}$$

for $i = 1, \mathcal{F}$. In other words, $q_i = q_{i\,0}$, for $i = 1, \mathcal{F}$, is an equilibrium state provided that all of the generalized forces, Q_i [see Eq. (10.8)], evaluated at $q_i = q_{i\,0}$, are zero. Let us suppose that this is the case.

12.3 Stability equations

It is clear that if our system is initialized in some equilibrium state, with all of the $\dot{q}_i$ set to zero, then it will remain in this state for ever. But what happens if the system is slightly perturbed from the equilibrium state?

Let

$$q_i = q_{i\,0} + \delta q_i, \tag{12.5}$$

for $i = 1, \mathcal{F}$, where the δq_i are *small*. To lowest order in δq_i, the kinetic energy (12.1) can be written

$$K \simeq \frac{1}{2} \sum_{i,j=1,\mathcal{F}} M_{ij}\, \delta\dot{q}_i\, \delta\dot{q}_j, \tag{12.6}$$

where

$$M_{ij} = m_{ij}(q_{1\,0}, q_{2\,0}, \cdots, q_{\mathcal{F}\,0}), \tag{12.7}$$

and

$$M_{ij} = M_{ji}. \tag{12.8}$$

Note that the weights M_{ij} in the quadratic form (12.6) are now *constants*.

Taylor expanding the potential energy function about the equilibrium state, up to second order in the δq_i, we obtain

$$U \simeq U_0 - \sum_{i=1,\mathcal{F}} Q_{i\,0}\, \delta q_i - \frac{1}{2} \sum_{i,j=1,\mathcal{F}} G_{ij}\, \delta q_i\, \delta q_j, \tag{12.9}$$

where $U_0 = U(q_{1\,0}, q_{2\,0}, \cdots, q_{\mathcal{F}\,0})$, the $Q_{i\,0}$ are specified in Eq. (12.4), and

$$G_{ij} = -\frac{\partial U(q_{1\,0}, q_{2\,0}, \cdots, q_{\mathcal{F}\,0})}{\partial q_i\, \partial q_j}. \tag{12.10}$$

Now, we can set U_0 to zero without loss of generality. Moreover, according to Eq. (12.4), the $Q_{i\,0}$ are all zero. Hence, the expression (12.9) reduces to

$$U \simeq -\frac{1}{2} \sum_{i,j=1,\mathcal{F}} G_{ij}\, \delta q_i\, \delta q_j. \tag{12.11}$$

Note that, since $\partial U/\partial q_i\, \partial q_j \equiv \partial U/\partial q_j\, \partial q_i$, the constants weights G_{ij} in the above quadratic form are invariant under interchange of the indices i and j: *i.e.*,

$$G_{ij} = G_{ji}. \tag{12.12}$$

With K and U specified by the quadratic forms (12.6) and (12.11), respectively, Lagrange's equations of motion (12.3) reduce to

$$\sum_{j=1,\mathcal{F}} (M_{ij}\, \delta\ddot{q}_j - G_{ij}\, \delta q_j) = 0, \tag{12.13}$$

for $i = 1, \mathcal{F}$. Note that the above coupled differential equations are *linear* in the δq_i. It follows that the solutions are *superposable*. Let us search for solutions of the above equations in which all of the perturbed coordinates δq_i have a common time variation of the form

$$\delta q_i(t) = \delta q_i\, \mathrm{e}^{\,\gamma\, t}, \tag{12.14}$$

for $i = 1, \mathcal{F}$. Now, Eqs. (12.13) are a set of $\mathcal{F}$ second-order differential equations. Hence, the most general solution contains $2\mathcal{F}$ arbitrary constants of integration. Thus, if we can find sufficient independent solutions of the form (12.14) to Eqs. (12.13) that the superposition of these solutions contains $2\mathcal{F}$ arbitrary constants then we can be sure that we have found the most general solution. Equations (12.13) and (12.14) yield

$$\sum_{j=1,\mathcal{F}} (G_{ij} - \gamma^2\, M_{ij})\, \delta q_j = 0, \tag{12.15}$$

which can be written more succinctly as a matrix equation:

$$(\mathbf{G} - \gamma^2\, \mathbf{M})\, \delta\mathbf{q} = \mathbf{0}. \tag{12.16}$$

Here, $\mathbf{G}$ is the real [see Eq. (12.10)], symmetric [see Eq. (12.12)], $\mathcal{F}\times\mathcal{F}$ matrix of the G_{ij} values. Furthermore, $\mathbf{M}$ is the real [see Eq. (12.1)], symmetric [see Eq. (12.8)], $\mathcal{F}\times\mathcal{F}$ matrix of the M_{ij} values. Finally, $\delta\mathbf{q}$ is the $1\times\mathcal{F}$ vector of the δq_i values, and $\mathbf{0}$ is a null vector.

12.4 Mathematical digression

Equation (12.16) takes the form of a matrix eigenvalue equation:

$$(\mathbf{G}-\lambda\,\mathbf{M})\,\mathbf{x}=\mathbf{0}. \tag{12.17}$$

Here, $\mathbf{G}$ and $\mathbf{M}$ are both real symmetric matrices, whereas λ is termed the *eigenvalue*, and $\mathbf{x}$ the associated *eigenvector*. The above matrix equation is essentially a set of $\mathcal{F}$ homogeneous simultaneous algebraic equations for the components of $\mathbf{x}$. As is well-known, a necessary condition for such a set of equations to possess a non-trivial solution is that the *determinant* of the matrix must be zero: *i.e.*,

$$|\mathbf{G}-\lambda\,\mathbf{M}|=0. \tag{12.18}$$

The above formula reduces to an $\mathcal{F}$th-order *polynomial* equation for λ. Hence, we conclude that Eq. (12.17) is satisfied by $\mathcal{F}$ eigenvalues, and $\mathcal{F}$ associated eigenvectors.

We can easily demonstrate that the eigenvalues are all *real*. Suppose that λ_k and $\mathbf{x}_k$ are the kth eigenvalue and eigenvector, respectively. Then we have

$$\mathbf{G}\,\mathbf{x}_k=\lambda_k\,\mathbf{M}\,\mathbf{x}_k. \tag{12.19}$$

Taking the transpose and complex conjugate of the above equation, and right multiplying by $\mathbf{x}_k$, we obtain

$$\mathbf{x}_k^{*T}\,\mathbf{G}^{*T}\,\mathbf{x}_k=\lambda_k^*\,\mathbf{x}_k^{*T}\,\mathbf{M}^{*T}\,\mathbf{x}_k. \tag{12.20}$$

Here, ^{T} denotes a transpose, and * a complex conjugate. However, since $\mathbf{G}$ and $\mathbf{M}$ are both real symmetric matrices, it follows that $\mathbf{G}^{*T}=\mathbf{G}$ and $\mathbf{M}^{*T}=\mathbf{M}$. Hence,

$$\mathbf{x}_k^{*T}\,\mathbf{G}\,\mathbf{x}_k=\lambda_k^*\,\mathbf{x}_k^{*T}\,\mathbf{M}\,\mathbf{x}_k. \tag{12.21}$$

Next, left multiplying Eq. (12.19) by $\mathbf{x}_k^{*T}$, we obtain

$$\mathbf{x}_k^{*T}\,\mathbf{G}\,\mathbf{x}_k = \lambda_k\,\mathbf{x}_k^{*T}\,\mathbf{M}\,\mathbf{x}_k. \tag{12.22}$$

Taking the difference between the above two expressions, we get

$$(\lambda_k^* - \lambda_k)\,\mathbf{x}_k^{*T}\,\mathbf{M}\,\mathbf{x}_k = 0. \tag{12.23}$$

Since $\mathbf{x}_k^{*T}\,\mathbf{M}\,\mathbf{x}_k$ is not generally zero, except in the trivial case where $\mathbf{x}_k$ is a null vector, we conclude that $\lambda_k^* = \lambda_k$ for all k. In other words, the eigenvalues are all real. It immediately follows that the eigenvectors can also be chosen to be all real.

Consider two distinct eigenvalues, λ_k and λ_l, with the associated eigenvectors $\mathbf{x}_k$ and $\mathbf{x}_l$, respectively. We have

$$\mathbf{G}\,\mathbf{x}_k = \lambda_k\,\mathbf{M}\,\mathbf{x}_k, \tag{12.24}$$

$$\mathbf{G}\,\mathbf{x}_l = \lambda_l\,\mathbf{M}\,\mathbf{x}_l. \tag{12.25}$$

Right multiplying the transpose of Eq. (12.24) by $\mathbf{x}_l$, and left multiplying Eq. (12.25) by $\mathbf{x}_k^T$, we obtain

$$\mathbf{x}_k^T\,\mathbf{G}\,\mathbf{x}_l = \lambda_k\,\mathbf{x}_k^T\,\mathbf{M}\,\mathbf{x}_l, \tag{12.26}$$

$$\mathbf{x}_k^T\,\mathbf{G}\,\mathbf{x}_l = \lambda_l\,\mathbf{x}_k^T\,\mathbf{M}\,\mathbf{x}_l. \tag{12.27}$$

Taking the difference between the above two expressions, we get

$$(\lambda_k - \lambda_l)\,\mathbf{x}_k^T\,\mathbf{M}\,\mathbf{x}_l = 0. \tag{12.28}$$

Hence, we conclude that

$$\mathbf{x}_k^T\,\mathbf{M}\,\mathbf{x}_l = 0, \tag{12.29}$$

provided $\lambda_k \neq \lambda_l$. In other words, two eigenvectors corresponding to two different eigenvalues are "orthogonal" to one another (in the sense specified in the above equation). Moreover, it is easily demonstrated that different eigenvectors corresponding to the same eigenvalue can be chosen in such a manner that they are also orthogonal to one another (see Sect. 9.5). Thus, we conclude that all of the eigenvectors are *mutually orthogonal*. Since Eq. (12.17) only specifies the directions, and not the lengths, of the eigenvectors, we are free to normalize our eigenvectors such that

$$\mathbf{x}_k^T\,\mathbf{M}\,\mathbf{x}_l = \delta_{kl}, \tag{12.30}$$

where $\delta_{kl} = 1$ when $k = l$, and $\delta_{kl} = 0$ otherwise. Note, finally, that since the $\mathbf{x}_k$, for $k = 1, \mathcal{F}$, are mutually orthogonal, they are *independent* (*i.e.*, one eigenvector cannot be expressed as a linear combination of the others), and completely span $\mathcal{F}$-dimensional vector space.

12.5 Normal modes

It follows from Eq. (12.14) and (12.15), plus the mathematical results contained in the previous subsection, that the most general solution to Eq. (12.13) can be written

$$\delta\mathbf{q}(t) = \sum_{k=1,\mathcal{F}} \delta\mathbf{q}_k(t), \tag{12.31}$$

where

$$\delta\mathbf{q}_k(t) = \left(\alpha_k \, \mathrm{e}^{+\sqrt{\lambda_k}\, t} + \beta_k \, \mathrm{e}^{-\sqrt{\lambda_k}\, t}\right) \mathbf{x}_k. \tag{12.32}$$

Here, the λ_k and the $\mathbf{x}_k$ are the eigenvalues and eigenvectors obtained by solving Eq. (12.17). Moreover, the α_k and β_k are arbitrary constants. Finally, we have made use of the fact that the two roots of $\gamma^2 = \lambda_k$ are $\gamma = \pm\sqrt{\lambda_k}$.

According to Eq. (12.31), the most general perturbed motion of the system consists of a *linear combination* of $\mathcal{F}$ different modes. These modes are generally termed *normal modes*, since they are mutually orthogonal (because the $\mathbf{x}_k$ are mutually orthogonal). Furthermore, it follows from the independence of the $\mathbf{x}_k$ that the normal modes are also *independent* (*i.e.*, one mode cannot be expressed as a linear combination of the others). The kth normal mode has a specific pattern of motion which is specified by the kth eigenvector, $\mathbf{x}_k$. Moreover, the kth mode has a specific time variation which is determined by the associated eigenvalue, λ_k. Recall that λ_k is *real*. Hence, there are only two possibilities. Either λ_k is *positive*, in which case we can write

$$\delta\mathbf{q}_k(t) = \left(\alpha_k \, \mathrm{e}^{+\gamma_k\, t} + \beta_k \, \mathrm{e}^{-\gamma_k\, t}\right) \mathbf{x}_k, \tag{12.33}$$

where $\lambda_k = \gamma_k^2$, or λ_k is *negative*, in which case we can write

$$\delta\mathbf{q}_k(t) = \left(\alpha_k \, \mathrm{e}^{+\mathrm{i}\,\omega_k\, t} + \beta_k \, \mathrm{e}^{-\mathrm{i}\,\omega_k\, t}\right) \mathbf{x}_k, \tag{12.34}$$

where $\lambda_k = -\omega_k^2$. In other words, if λ_k is positive then the kth normal mode *grows* or decays secularly in time, whereas if λ_k is negative then the kth normal mode *oscillates* in time. Obviously, if the system possesses one or more normal modes which grow secularly in time then the equilibrium about which we originally expanded the equations of motion must be an *unstable* equilibrium. On the other hand, if all of the normal modes oscillate in time then the equilibrium is *stable*. Thus, we conclude that whilst Eq. (12.4) is the condition for the existence of an equilibrium state in a many degree of freedom system, the condition for the equilibrium to be *stable* is that all of the eigenvalues of the stability equation (12.17) must be *negative*.

The arbitrary constants α_k and β_k appearing in expression (12.32) are determined from the *initial conditions*. Thus, if $\delta\mathbf{q}^{(0)} = \delta\mathbf{q}(t=0)$ and $\delta\dot{\mathbf{q}}^{(0)} = \delta\dot{\mathbf{q}}(t=0)$ then it is easily demonstrated from Eqs. (12.30)–(12.32) that

$$\mathbf{x}_k^T\,\mathbf{M}\,\delta\mathbf{q}^{(0)} = \alpha_k + \beta_k, \tag{12.35}$$

and

$$\mathbf{x}_k^T\,\mathbf{M}\,\delta\dot{\mathbf{q}}^{(0)} = \sqrt{\lambda_k}(\alpha_k - \beta_k). \tag{12.36}$$

Hence,

$$\alpha_k = \frac{\mathbf{x}_k^T\,\mathbf{M}\,\delta\mathbf{q}^{(0)} + \mathbf{x}_k^T\,\mathbf{M}\,\delta\dot{\mathbf{q}}^{(0)}/\sqrt{\lambda_k}}{2}, \tag{12.37}$$

$$\beta_k = \frac{\mathbf{x}_k^T\,\mathbf{M}\,\delta\mathbf{q}^{(0)} - \mathbf{x}_k^T\,\mathbf{M}\,\delta\dot{\mathbf{q}}^{(0)}/\sqrt{\lambda_k}}{2}. \tag{12.38}$$

Note, finally, that since there are $2\mathcal{F}$ arbitrary constants (two for each of the $\mathcal{F}$ normal modes), we can be sure that Eq. (12.31) represents the most general solution to Eq. (12.13).

12.6 Normal coordinates

Since the eigenvectors $\mathbf{x}_k$, for $k = 1, \mathcal{F}$, span $\mathcal{F}$-dimensional vector space, we can always write the displacement vector $\delta\mathbf{q}$ as some linear combination of the $\mathbf{x}_k$: *i.e.*,

$$\delta\mathbf{q}(t) = \sum_{k=1,\mathcal{F}} \eta_k(t)\,\mathbf{x}_k. \tag{12.39}$$

We can regard the $\eta_k(t)$ as a new set of generalized coordinates, since specifying the η_k is equivalent to specifying the δq_k (and, hence, the q_k). The η_k are usually termed *normal coordinates*. According to Eqs. (12.30) and (12.39), the normal coordinates can be written in terms of the δq_k as

$$\eta_k = \mathbf{x}_k^T \, \mathbf{M} \, \delta\mathbf{q}. \tag{12.40}$$

Let us now try to express K, U, and the equations of motion in terms of the η_k.

The kinetic energy can be written

$$K = \frac{\gamma^2}{2} \, \delta\mathbf{q}^T \, \mathbf{M} \, \delta\mathbf{q}, \tag{12.41}$$

where use has been made of Eqs. (12.6) and (12.14), It follows from (12.39) that

$$K = \frac{\gamma^2}{2} \sum_{k,l=1,\mathcal{F}} \eta_k \, \eta_l \, \mathbf{x}_k^T \, \mathbf{M} \, \mathbf{x}_l. \tag{12.42}$$

Finally, making use of the orthonormality condition (12.30), we obtain

$$K = \frac{\gamma^2}{2} \sum_{k=1,\mathcal{F}} \eta_k^2, \tag{12.43}$$

or

$$K = \frac{1}{2} \sum_{k=1,\mathcal{F}} \dot{\eta}_k^2. \tag{12.44}$$

Hence, the kinetic energy K takes the form of a *diagonal* quadratic form when expressed in terms of the normal coordinates.

The potential energy can be written

$$U = -\frac{1}{2} \, \delta\mathbf{q}^T \, \mathbf{G} \, \delta\mathbf{q}, \tag{12.45}$$

where use has been made of Eqs. (12.11). It follows from (12.39) that

$$U = -\frac{1}{2} \sum_{k,l=1,\mathcal{F}} \eta_k \, \eta_l \, \mathbf{x}_k^T \, \mathbf{G} \, \mathbf{x}_l. \tag{12.46}$$

Finally, making use of Eq. (12.19) and the orthonormality condition (12.30), we obtain

$$U = -\frac{1}{2}\sum_{k=1,\mathcal{F}} \lambda_k\,\eta_k^2. \tag{12.47}$$

Hence, the potential energy U also takes the form of a *diagonal* quadratic form when expressed in terms of the normal coordinates.

Writing Lagrange's equations of motion in terms of the normal coordinates, we obtain [*cf.*, Eq. (12.3)]

$$\frac{d}{dt}\left(\frac{\partial K}{\partial \dot{\eta}_k}\right) - \frac{\partial K}{\partial \eta_k} + \frac{\partial U}{\partial \eta_k} = 0, \tag{12.48}$$

for $k = 1, \mathcal{F}$. Thus, it follows from Eqs. (12.44) and (12.47) that

$$\ddot{\eta}_k = \lambda_k\,\eta_k, \tag{12.49}$$

for $k = 1, \mathcal{F}$. In other words, Lagrange's equations reduce to a set of $\mathcal{F}$ *uncoupled* simple harmonic equations when expressed in terms of the normal coordinates. The solutions to the above equations are obvious: *i.e.*,

$$\eta_k(t) = \alpha_k\,\mathrm{e}^{+\sqrt{\lambda_k}\,t} + \beta_k\,\mathrm{e}^{-\sqrt{\lambda_k}\,t}, \tag{12.50}$$

where α_k and β_k are arbitrary constants. Hence, it is clear from Eqs. (12.39) and (12.50) that the most general solution to the perturbed equations of motion is indeed given by Eqs. (12.31) and (12.32).

In conclusion, the perturbed equations of motion of a many degree of freedom dynamical system take a particularly simple form when expressed in terms of the normal coordinates. Each normal coordinate specifies the instantaneous displacement of an independent mode of oscillation (or secular growth) of the system. Moreover, each normal coordinate oscillates at a characteristic frequency (or grows at a characteristic rate), and is completely unaffected by the other coordinates.

12.7 Spring-coupled masses

Consider the two degree of freedom dynamical system pictured in Fig. 49. In this system, two point objects of mass m are free to move in one dimension. Further-

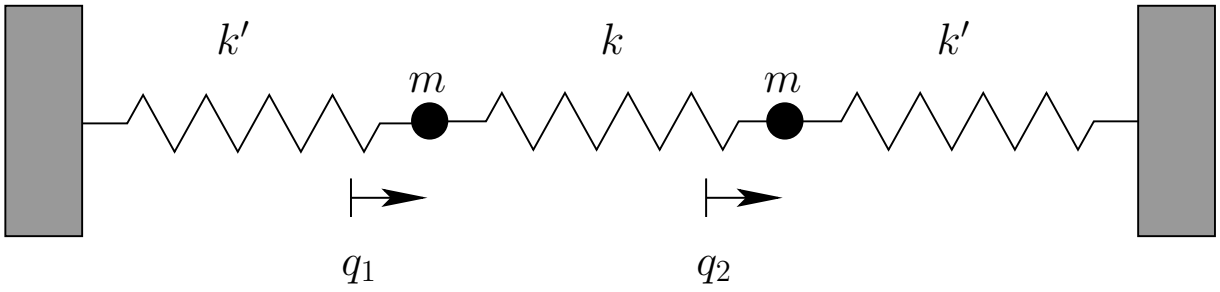

Figure 49:

more, the masses are connected together by a spring of spring constant k, and are also each attached to fixed supports via springs of spring constant k'.

Let q_1 and q_2 be the displacements of the first and second masses, respectively, from the equilibrium state. If follows that the extensions of the left-hand, middle, and right-hand springs are q_1, $q_2 - q_1$, and $-q_2$, respectively. The kinetic energy of the system takes the form

$$K = \frac{m}{2}\,(\dot{q}_1^{\,2} + \dot{q}_2^{\,2}), \tag{12.51}$$

whereas the potential energy is written

$$U = \frac{1}{2}\left[k'\,q_1^{\,2} + k\,(q_2 - q_1)^2 + k'\,q_2^{\,2}\right]. \tag{12.52}$$

The above expression can be rearranged to give

$$U = \frac{1}{2}\left[(k + k')\,q_1^{\,2} - 2\,k\,q_1\,q_2 + (k + k')\,q_2^{\,2}\right]. \tag{12.53}$$

A comparison of Eqs. (12.51) and (12.53) with the standard forms (12.6) and (12.11) yields the following expressions for the mass matrix $\mathbf{M}$ and the force matrix $\mathbf{G}$:

$$\mathbf{M} = \begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix}, \tag{12.54}$$

$$\mathbf{G} = \begin{pmatrix} -k - k' & k \\ k & -k - k' \end{pmatrix}. \tag{12.55}$$

Now, the equation of motion of the system takes the form [see Eq. (12.17)]

$$(\mathbf{G} - \lambda\,\mathbf{M})\,\mathbf{x} = \mathbf{0}, \tag{12.56}$$

where $\mathbf{x}$ is the column vector of the q_1 and q_2 values. The solubility condition for the above equation is

$$|\mathbf{G} - \lambda\,\mathbf{M}| = 0, \tag{12.57}$$

or

$$\begin{vmatrix} -k - k' - \lambda\,m & k \\ k & -k - k' - \lambda\,m \end{vmatrix} = 0, \tag{12.58}$$

which yields the following quadratic equation for the eigenvalue λ:

$$m^2\,\lambda^2 + 2\,m\,(k + k')\,\lambda + k'\,(k' + 2\,k) = 0. \tag{12.59}$$

The two roots of the above equation are

$$\lambda_1 = -\frac{k'}{m}, \tag{12.60}$$

$$\lambda_2 = -\frac{(2\,k + k')}{m}. \tag{12.61}$$

The fact that the roots are negative implies that both normal modes are *oscillatory* in nature: *i.e.*, the original equilibrium is *stable*. The characteristic oscillation frequencies of the modes are

$$\omega_1 = \sqrt{-\lambda_1} = \sqrt{\frac{k'}{m}}, \tag{12.62}$$

$$\omega_2 = \sqrt{-\lambda_2} = \sqrt{\frac{2\,k + k'}{m}}. \tag{12.63}$$

Now, the first row of Eq. (12.56) gives

$$\frac{q_1}{q_2} = \frac{k}{k + k' + \lambda\,m}. \tag{12.64}$$

Moreover, Eqs. (12.30) and (12.54) yield the following normalization condition for the eigenvectors:

$$\mathbf{x}_k^T\,\mathbf{x}_k = m^{-1}, \tag{12.65}$$

for $k = 1, 2$. It follows that the two eigenvectors are

$$\mathbf{x}_1 = (2\,m)^{-1/2}\,(1,\quad 1), \tag{12.66}$$

$$\mathbf{x}_2 = (2\,m)^{-1/2}\,(1, -1). \tag{12.67}$$

According to Eqs. (12.62)–(12.63) and (12.66)–(12.67), our two degree of freedom system possesses two normal modes. The first mode oscillates at the frequency ω_1, and is a purely *symmetric* mode: *i.e.*, $q_1 = q_2$. Note that such a mode does not stretch the middle spring. Hence, ω_1 is independent of k. In fact, ω_1 is simply the characteristic oscillation frequency of a mass m on the end of a spring of spring constant k'. The second mode oscillates at the frequency ω_2, and is a purely *anti-symmetric* mode: *i.e.*, $q_1 = -q_2$. Since such a mode stretches the middle spring, the second mode experiences a greater restoring force than the first, and hence has a higher oscillation frequency: *i.e.*, $\omega_2 > \omega_1$.

Note, finally, from Eqs. (12.40) and (12.54), that the normal coordinates of the system are:

$$\eta_1 = \sqrt{\frac{m}{2}}\,(q_1 + q_2), \tag{12.68}$$

$$\eta_2 = \sqrt{\frac{m}{2}}\,(q_1 - q_2). \tag{12.69}$$

When expressed in terms of these normal coordinates, the kinetic and potential energies of the system reduce to

$$K = \frac{1}{2}\,(\dot{\eta}_1^{\,2} + \dot{\eta}_2^{\,2}), \tag{12.70}$$

$$U = \frac{1}{2}\,(\omega_1^2\,\eta_1^2 + \omega_2^2\,\eta_2^2), \tag{12.71}$$

respectively.

12.8 Triatomic molecule

Consider the simple model of a linear triatomic molecule (*e.g.*, carbon dioxide) illustrated in Fig. 50. The molecule consists of a central atom of mass M flanked by two identical atoms of mass m. The atomic bonds are represented as springs of spring constant k. The linear displacements of the flanking atoms are q_1 and q_2, whilst that of the central atom is q_3. Let us investigate the linear modes of oscillation our model molecule.

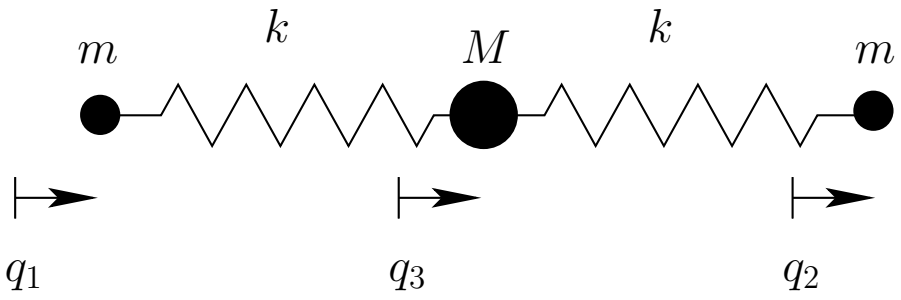

Figure 50:

The kinetic energy of the molecule is written

$$K = \frac{m}{2}\,(\dot{q}_1^{\,2} + \dot{q}_2^{\,2}) + \frac{M}{2}\,\dot{q}_3^{\,2}, \tag{12.72}$$

whereas the potential energy takes the form

$$U = \frac{k}{2}\,(q_3 - q_1)^2 + \frac{k}{2}\,(q_2 - q_3)^2. \tag{12.73}$$

Clearly, we have a three degree of freedom dynamical system. However, we can reduce this to a two degree of freedom system by only considering *oscillatory* modes of motion, and, hence, neglecting *translational* modes. We can achieve this by demanding that the center of mass of the system remains stationary. In other words, we require that

$$m\,(q_1 + q_2) + M\,q_3 = 0. \tag{12.74}$$

This constraint can be rearranged to give

$$q_3 = -\frac{m}{M}\,(q_1 + q_2). \tag{12.75}$$

Eliminating q_3 from Eqs. (12.72) and (12.73), we obtain

$$K = \frac{m}{2}\left[(1+\alpha)\,\dot{q}_1^{\,2} + 2\,\alpha\,\dot{q}_1\,\dot{q}_2 + (1+\alpha)\,\dot{q}_2^{\,2}\right], \tag{12.76}$$

and

$$U = \frac{k}{2}\left[(1 + 2\,\alpha + 2\,\alpha^2)\,q_1^2 + 4\,\alpha\,(1+\alpha)\,q_1\,q_2 + (1 + 2\,\alpha + 2\,\alpha^2)\,q_2^2\right], \tag{12.77}$$

respectively, where $\alpha = m/M$.

A comparison of the above expressions with the standard forms (12.6) and (12.11) yields the following expressions for the mass matrix $\mathbf{M}$ and the force matrix $\mathbf{G}$:

$$\mathbf{M} = m \begin{pmatrix} 1+\alpha & \alpha \\ \alpha & 1+\alpha \end{pmatrix}, \tag{12.78}$$

$$\mathbf{G} = -k \begin{pmatrix} 1+2\,\alpha+2\,\alpha^2 & 2\,\alpha\,(1+\alpha) \\ 2\,\alpha\,(1+\alpha) & 1+2\,\alpha+2\,\alpha^2 \end{pmatrix}. \tag{12.79}$$

Now, the equation of motion of the system takes the form [see Eq. (12.17)]

$$(\mathbf{G}-\lambda\,\mathbf{M})\,\mathbf{x} = \mathbf{0}, \tag{12.80}$$

where $\mathbf{x}$ is the column vector of the q_1 and q_2 values. The solubility condition for the above equation is

$$|\mathbf{G}-\lambda\,\mathbf{M}| = 0, \tag{12.81}$$

which yields the following quadratic equation for the eigenvalue λ:

$$(1+2\,\alpha)\,\left[m^2\,\lambda^2+2\,m\,k\,(1+\alpha)\,\lambda+k^2\,(1+2\alpha)\right] = 0. \tag{12.82}$$

The two roots of the above equation are

$$\lambda_1 = -\frac{k}{m}, \tag{12.83}$$

$$\lambda_2 = -\frac{k\,(1+2\alpha)}{m}. \tag{12.84}$$

The fact that the roots are negative implies that both normal modes are indeed *oscillatory* in nature. The characteristic oscillation frequencies are

$$\omega_1 = \sqrt{-\lambda_1} = \sqrt{\frac{k}{m}}, \tag{12.85}$$

$$\omega_2 = \sqrt{-\lambda_2} = \sqrt{\frac{k\,(1+2\,\alpha)}{m}}. \tag{12.86}$$

Equation (12.80) can now be solved, subject to the normalization condition (12.30), to give the two eigenvectors:

$$\mathbf{x}_1 = (2\,m)^{-1/2}\,(1,-1), \tag{12.87}$$

$$\mathbf{x}_2 = (2\,m)^{-1/2}\,(1+2\,\alpha)^{-1/2}\,(1,\quad 1). \tag{12.88}$$

Thus, we conclude from Eqs. (12.75) and (12.85)–(12.88) that our model molecule possesses two normal modes of oscillation. The first mode oscillates at the frequency ω_1, and is an *anti-symmetric* mode in which $q_1 = -q_2$ and $q_3 = 0$. In other words, in this mode of oscillation, the two end atoms move in opposite directions whilst the central atom remains stationary. The second mode oscillates at the frequency ω_2, and is a mixed symmetry mode in which $q_1 = q_2$ but $q_3 = -2\,\alpha\,q_1$. In other words, in this mode of oscillation, the two end atoms move in the same direction whilst the central atom moves in the opposite direction.

Finally, it is easily demonstrated that the normal coordinates of the system are

$$\eta_1 = \sqrt{\frac{m}{2}}\,(q_1 - q_2), \tag{12.89}$$

$$\eta_2 = \sqrt{\frac{m\,(1+2\,\alpha)}{k}}\,(q_1 + q_2). \tag{12.90}$$

When expressed in terms of these coordinates, K and U reduce to

$$K = \frac{1}{2}\,(\dot{\eta}_1^{\,2} + \dot{\eta}_2^{\,2}), \tag{12.91}$$

$$U = \frac{1}{2}\,(\omega_1^2\,\eta_1^{\,2} + \omega_2^2\,\eta_2^{\,2}), \tag{12.92}$$

respectively.